

# The classical limit of non-integrable quantum systems

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The classical limit of non-integrable quantum systems is studied. We define non-integrable quantum systems as those which have, as their classical limit, a non-integrable classical system. In order to obtain this limit, the *self-induced decoherence* approach and the corresponding classical limit are generalized from integrable to non-integrable systems. In this approach, the lost of information, usually conceived as the result of a coarse-graining or the trace of an environment, is produced by a particular choice of the algebra of observables and the systematic use of mean values, that project the unitary evolution onto an effective non-unitary one. The decoherence times computed with this approach coincide with those of the literature. By means of our method, we can obtain the classical limit of the quantum state of a non-integrable system, which turns out to be a set of unstable, potentially chaotic classical trajectories contained in the Wigner transformation of the quantum state.

PACS number(s) 03.65.Bz

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## I. INTRODUCTION

The problem of the classical limit of quantum mechanics has a long history. In the beginning, on the basis of the analogy with special relativity where the limit  $c \rightarrow \infty$  leads to the classical behavior, it was thought that the classical limit was just the limit  $\hbar \rightarrow 0$ . But it was soon realized that this was only one element of the problem, namely, *macroscopicity*, and that other elements must be taken into account: e.g. quantum mechanics has a probabilistic non-Boolean structure while classical mechanics has a non-probabilistic and Boolean one. Thus, necessarily two new elements must come into play:

- *Decoherence*, that transforms the non-Boolean structure into a Boolean one, and
- *Localization (actualization or the choice of a trajectory)* that, with macroscopicity -which circumvents the uncertainty principle- turns the probabilistic structure into a non-probabilistic one.

In general, decoherence in quantum systems is defined as a process that leads to the diagonalization of a density matrix (more precisely, to the decay of the cross-terms in the expectation value of an observable in some basis). In a first period, decoherence was explained as the result of the destructive interference of the off-diagonal elements of the density matrix (see [1], [2]); however, this line of research was abandoned due to technical difficulties derived from the formalism used to describe the process. As a consequence, decoherence began to be conceived as produced by the interaction between a system and its environment. This approach gave rise to the einselection program, based on the works of Zeh ([3], [4], [5]) and later developed by Zurek and coworkers ([6], [7], [8], [9], [10], [11], [12]). Although many relevant results have been obtained by means of einselection, this approach still involves certain unsolved problems, as those related with the explanation of the emergence of classicality in closed quantum systems, the criterion for introducing the 'cut' between the system and its environment, and the definition of the preferred ('pointer') basis where the system behaves classically (see [13], [14]). As the result of these and other difficulties, a number of alternative accounts of decoherence have been proposed (see [15], [16], [17], [18], [19], [20]).

In a series of papers ([21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [13]) we have returned to the initial idea of the destructive interference of the off-diagonal terms of the density matrix, but now on the basis of a different formalism: the formalism introduced by van Hove ([36], [37], [38], [39], [40]). We have called this new approach '*self-induced decoherence*' [13] because, from this viewpoint, decoherence is not produced by the interaction between a system and its environment, but results from the own dynamics of the whole quantum system governed by a Hamiltonian with continuous spectrum. In this approach, the difficulties derived from the einselection program are absent: self-induced decoherence can be used in closed systems as the universe [30], the definition of a convenient

subalgebra plays the role of the coarse-graining induced by the environment, avoiding the 'cut' problem [13], and the pointer basis is perfectly defined (see [13], [34] and Section III.C below).

Self-induced decoherence is capable of addressing relevant problems from a general perspective, e.g. the problem of supplying a good definition of the classical limit *in all cases*<sup>1</sup>. Let us explain the essence of the idea supporting this new approach. When we deal with continuous spectra, the destructive interference is embodied in the Riemann-Lebesgue theorem which states that, if  $f(\nu) \in \mathbb{L}_1$ , then

$$\lim_{t \rightarrow \infty} \int_{-a}^a f(\nu) e^{-i \frac{\nu t}{\hbar}} d\nu = 0$$

where  $e^{-i \frac{\nu t}{\hbar}}$  is the  $\nu$ -oscillating factor that produces the destructive interference. In the case of decoherence,  $\nu = \omega - \omega'$ , where  $\omega, \omega'$  are the continuous indices of the density operator  $\rho(\omega, \omega')$  in the energy eigenbasis; then,  $\nu = 0$  corresponds to the diagonal. However, to require that  $f(\nu) \in \mathbb{L}_1$  is to ask too much regularity to function  $f(\nu)$ , because in this case not only the off-diagonal ( $\nu \neq 0$ ) terms, but also the diagonal ( $\nu = 0$ ) ones will vanish when  $t \rightarrow \infty$ . Therefore, we use less regular functions, precisely  $f(\nu) = A\delta(\nu) + f_1(\nu)$ , where only  $f_1(\nu) \in \mathbb{L}_1$ . In this case,

$$\lim_{t \rightarrow \infty} \int_{-a}^a f(\nu) e^{-i \frac{\nu t}{\hbar}} d\nu = A$$

and the diagonal terms  $A\delta(\nu)$  remain while the off-diagonal terms  $f_1(\nu)$  vanish. We will apply this main idea to the *non-integrable case*, and present the computations in all detail in Section III.B, by using our previous results on quantum systems with continuous spectrum contained in papers [22], [23], [24], [27], [28] and [29]. With this strategy we have already obtained, in paper [26], the classical limit for *integrable* systems. We have also presented this result in more rigorous mathematical basis in [33] and explained the physical foundations of the method in papers [13] and [34]. The mathematical basis of the theory is explained in papers [41] and [32]. But, of course, the big challenge to prove the consistency and generality of the method is to find its version for *non-integrable* systems, obtaining unstable, potentially chaotic classical trajectories as a final result, which could explain models as those of ref. [42].

In the case of integrable systems, the classical limit was obtained by a combination of the van Hove formalism and the Weyl-Wigner-Moyal isomorphism in a globally defined pointer basis. But in the non-integrable case, such a global basis does not exist. Nevertheless, the just quoted isomorphism is what allows us to relax the global condition and to generalize the formalism: quantum mechanics is formulated in a phase space that is covered with charts where *local pointer bases can be defined*. The set of all these local pointer bases will yield decomposition (38), which is the essential tool of this paper.

The formalism of the theory is presented in a self-comprehensive way, with a mathematics as simple as possible and in the simplest possible case; this seems enough for the physical purposes of this paper. In Section II, a brief review of the Weyl-Wigner-Moyal mapping is developed, and in Section III, the theory of decoherence in non-integrable systems is explained. In Section IV, the classical limit of quantum non-integrable system is obtained. In Section V, the localization phenomena is briefly discussed, and in Section VI, our previous results are generalized to the case of partially non-integrable systems. In the conclusion (Section VII), we list the possible future applications of the theory and explain why it could be considered as a *minimal formalism for quantum chaos*. Finally, in Appendix A we explain the integration of two systems of differential equations relevant to our theory, and in Appendix B we give an example of non-integrable system.

## II. WEYL-WIGNER-MOYAL MAPPING

Let  $\mathcal{M} = \mathcal{M}_{2(N+1)} \equiv \mathbb{R}^{2(N+1)}$  be the phase space of our classical system. The functions over this phase space will be called  $f(\phi)$ , where  $\phi$  symbolizes the coordinates over  $\mathcal{M}$

$$\phi^a = (q^1, \dots, q^{N+1}, p_q^1, \dots, p_q^{N+1}) \quad a = 1, 2, \dots, 2(N+1) \quad (1)$$

As it is known (see [43], [44]), we can map  $\hat{\mathcal{A}}$ , the algebra of regular operators  $\hat{O}$  of our quantum system, on  $\mathcal{A}_q$ , the algebra of integrable functions over  $\mathcal{M}$ , via the *Wigner symbol*

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<sup>1</sup>Precisely: in all cases where the system do have a classical limit; e.g., systems with no quasi-continuous limit, yielding to a non-continuous energy spectrum, are excluded [35].

$$\text{ symb} : \widehat{\mathcal{A}} \rightarrow \mathcal{A}_q \quad \text{ symb}\widehat{O} = O(\phi) \quad (2)$$

Precisely: let us consider that  $\mathcal{M}$  has a symplectic form

$$\omega_{ab} = \begin{pmatrix} 0 & I_{N+1} \\ -I_{N+1} & 0 \end{pmatrix} \quad \omega^{ab} = \begin{pmatrix} 0 & -I_{N+1} \\ I_{N+1} & 0 \end{pmatrix} \quad (3)$$

Then,

$$\text{ symb}\widehat{f} \doteq f(\phi) = \int d^{2(N+1)}\psi \exp\left(\frac{i}{\hbar}\psi^a\omega_{ab}\psi^b\right) Tr\left(\widehat{T}(\psi)\widehat{f}\right) \quad (4)$$

where  $\widehat{f} \in \widehat{\mathcal{A}}$ ,  $f(\phi) \in \mathcal{A}_q$ , and

$$\widehat{T}(\psi) = \exp\left(\frac{i}{\hbar}\psi^a\omega_{ab}\widehat{\phi}^b\right) \quad (5)$$

On  $\mathcal{A}_q$  we can define the *star product* (i.e. the classical operator related with the multiplication on  $\widehat{\mathcal{A}}$  and, therefore, defining the corresponding operation on  $\mathcal{A}_q$ ) as

$$\text{ symb}(\widehat{f}\widehat{g}) = \text{ symb}\widehat{f} * \text{ symb}\widehat{g} = (f * g)(\phi) \quad (6)$$

It can be proved ([43], eq.(2.59)) that

$$(f * g)(\phi) = f(\phi) \exp\left(-\frac{i\hbar}{2}\overleftarrow{\partial}_a\omega^{ab}\overrightarrow{\partial}_b\right) g(\phi) = g(\phi) \exp\left(\frac{i\hbar}{2}\overleftarrow{\partial}_a\omega^{ab}\overrightarrow{\partial}_b\right) f(\phi) \quad (7)$$

We also define the *Moyal bracket* as the symbol corresponding to the commutator in  $\widehat{\mathcal{A}}$

$$\{f, g\}_{mb} = \frac{1}{i\hbar}(f * g - g * f) = \text{ symb}\left(\frac{1}{i\hbar}[f, g]\right) = \frac{1}{i\hbar}f(\phi) \sin\left(-\frac{i\hbar}{2}\overleftarrow{\partial}_a\omega^{ab}\overrightarrow{\partial}_b\right) g(\phi) \quad (8)$$

In the limit  $\hbar \rightarrow 0$ , the star product becomes the ordinary product, and the Moyal bracket becomes the Poisson bracket<sup>2</sup>

$$(f * g)(\phi) = f(\phi)g(\phi) + 0(\hbar) \quad (9)$$

$$\{f, g\}_{mb} = \{f, g\}_{pb} + 0(\hbar^2) \quad (10)$$

Then, we can either say that when  $\hbar \rightarrow 0$  the quantum structure tends to the classical one, or that when  $\hbar \neq 0$  the classical structure is *quantized* or *deformed* into the quantum one.

Let us observe that if  $\widehat{f}$  commutes with  $\widehat{g}$ , eqs.(7) and (9) change to

$$(f * g)(\phi) = f(\phi) \cos\left(-\frac{i\hbar}{2}\overleftarrow{\partial}_a\omega^{ab}\overrightarrow{\partial}_b\right) g(\phi) \quad (11)$$

$$(f * g)(\phi) = f(\phi)g(\phi) + 0(\hbar^2) \quad (12)$$

as it can be proved using eq.(7).

Finally, if we want that the mapping *symb* be one-to-one, we must define a unique inverse of *symb*, namely, the usual quantization rule  $q \rightarrow \widehat{q}$ ,  $p \rightarrow \widehat{p}$  endowed with a unique ordering prescription, e.g. the symmetrical or *Weyl ordering prescription* that maps

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<sup>2</sup>From eq. (7) it is clear that these  $0(\hbar)$  are continuous functions in the limit  $\hbar = 0$ . This fact will be important in Section V.

$$\text{ symb}^{-1}(qp) = \frac{1}{2}(\widehat{q}\widehat{p} + \widehat{p}\widehat{q}) \quad (13)$$

Then, we have

$$\text{ symb}^{-1} : \mathcal{A}_q \rightarrow \widehat{\mathcal{A}}, \quad \text{ symb} : \widehat{\mathcal{A}} \rightarrow \mathcal{A}_q \quad (14)$$

The one-to-one mapping so defined is the *Weyl-Wigner-Moyal symbol*. With  $\text{ symb}^{-1}$ , we can 'deform' the classical system and obtain a quantum mechanical system. With  $\text{ symb}$  we go from usual quantum mechanics to a quantum mechanics 'alla classica', formulated over a phase space  $\mathcal{M}$ , that will become the usual classical picture in the limit  $\hbar \rightarrow 0$  (as we will explain below in detail). The relation between the two structures, given by eq.(14) (and eq.(66) below), is an isomorphism that we will call *Weyl-Wigner-Moyal isomorphism*, the only one we will use in this paper.

Since  $\widehat{\mathcal{A}}$  is a space of operators on a Hilbert space  $\mathcal{H}$ , so it is its dual  $\widehat{\mathcal{A}}'$ ; then, as it is known, the symbol for any  $\widehat{\rho} \in \widehat{\mathcal{A}}'$  is defined as<sup>3</sup>

$$\rho(\phi) = \text{ symb}\widehat{\rho} = (2\pi\hbar)^{-(N+1)} \text{ symb}_{(\text{for operators})}\widehat{\rho} \quad (15)$$

where the  $\text{ symb}$  for operators is defined by the eqs.(4) and (5). From this definition, we have (see [43], eq.(2.13))

$$(\widehat{\rho}|\widehat{O}) = (\text{ symb}\widehat{\rho}|\text{ symb}\widehat{O}) = \int d\phi^{2(N+1)} \rho(\phi) O(\phi) \quad (16)$$

and in  $\widehat{\mathcal{A}}$  and  $\widehat{\mathcal{A}}'$  all the equations are the usual ones (i.e. those of papers [43] and [44]). Let us remark that the last equation is the cornerstone of our theory of the classical limit. In fact, as we will see, *it will remain the same when we go from regular to singular objects*. Once this statement is understood, the translation from the quantum language to the classical one will be easy.

### III. DECOHERENCE IN NON-INTEGRABLE SYSTEMS

#### A. Local CSCO

a.- We will begin with demonstrating an important theorem: when our quantum system is endowed with a CSCO of  $N + 1$  observables containing  $\widehat{H}$  that defines a basis in terms of which the state of the system can be expressed, the underlying classical system is *integrable*. In fact, let a classical system be defined in a phase space  $\mathcal{M} \equiv \mathbb{R}^{2(N+1)}$  that can be deformed 'alla Weyl'. If our quantum system is endowed with a  $N + 1$ -CSCO  $\{\widehat{H}, \widehat{O}_1, \dots, \widehat{O}_N\}$ , the Moyal brackets of these quantities are

$$\{O_I(\phi), O_J(\phi)\}_{mb} = \text{ symb} \left( \frac{1}{i\hbar} [\widehat{O}_I, \widehat{O}_J] \right) = 0 \quad (17)$$

where  $I, J, \dots = 0, 1, \dots, N$  and  $\widehat{H} = \widehat{O}_0$ . Then, when  $\hbar \rightarrow 0$ , from eq.(10) we know that

$$\{O_I(\phi), O_J(\phi)\}_{pb} = 0 \quad (18)$$

Thus, as  $H(\phi) = O_0(\phi)$ , the set  $\{O_I(\phi)\}$  is a complete set of  $N + 1$  constants of the motion in involution, globally defined over all  $\mathcal{M}$  and, therefore, the system is integrable. q. e. d.

As a consequence, *non-integrable* classical systems (precisely classical and also macroscopic ones such that  $\hbar \approx 0$ ), in their quantum version, cannot have a CSCO of  $N + 1$  observables globally defined containing  $\widehat{H}$ . But, according to the self-induced approach, the pointer basis is precisely the eigenbasis a global  $N + 1$ -CSCO (in such a way that the vectors of the pointer basis turn out to be stationary states, see [26]). Therefore, pointer bases cannot be

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<sup>3</sup>In the case of states, we must add a new factor  $(2\pi\hbar)^{-(N+1)}$  to definition (4) in order to preserve the usual normalization of  $\rho(\phi)$ . However,  $\rho(\phi)$  is not non-negatively defined. With decoherence and  $\hbar \rightarrow 0$  we will obtain a non-negatively defined  $\rho(\phi)$ , and  $\mathcal{A}_q \rightarrow \mathcal{A}$ , the classical boolean algebra of  $\mathbb{L}_1$  operators over  $\mathcal{M}$ .

globally defined in non-integrable systems. These systems can be adequately quantized, but it is impossible (at least globally) to define a complete stationary eigenbasis of  $N + 1$ -CSCO and, a fortiori, a pointer  $N + 1$ -CSCO or a pointer basis where the system would decohere according to the self-induced approach.<sup>4</sup> This is the main problem with non-integrable quantum systems.<sup>5</sup>

b.- We will now prove that  $N + 1$  constants of the motion in involution always exist locally.<sup>6</sup> Let us consider a non-integrable quantum system (i.e. with no global  $N + 1$ -CSCO), but let us suppose that, as usual,  $H(\phi) = \text{symp} \hat{H}$  is globally defined over  $\mathcal{M}$  (this means that any non-global CSCO has at least one global observable:  $\hat{H}$ ).<sup>7</sup> Now we can try to find  $N$  constants of the motion  $\{O_I(\phi)\}$  ( $I = 1, 2, \dots, N$ ) satisfying

$$\{H(\phi), O_I(\phi)\}_{pb} = \sum_{j=1}^N \frac{\partial H}{\partial p_{qj}} \frac{\partial O_I}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial O_I}{\partial p_{qj}} = 0 \quad (19)$$

This is a system of  $N$  partial differential equations which, with adequate boundary conditions, has a unique solution in a *maximal domain of integration*  $\mathcal{D}_{\phi_i}$  around any point  $\phi_i \in \mathcal{M}$  (provided that the functions involved satisfy reasonable -e.g. Lipschitzian- mathematical conditions that we assume<sup>8</sup>).

But we would like to obtain a set of constants of the motion in *involution*. Then, let us suppose that  $N$  different initial conditions for eq.(19) are given in a  $2N + 1$  dimensional hypersurface containing  $\phi_i$ , that we will call  $\mathcal{D}_{\phi_i}^N$ . Integrating (19) we will obtain  $N$  constants of the motion  $O_I(\phi)$ . Moreover, we can easily show that, if these solutions are in involution in  $\mathcal{D}_{\phi_i}^N$ , they will remain in involution in the domain  $\mathcal{D}_{\phi_i} = \mathcal{D}_{\phi_i}^{N+1}$  of  $2(N + 1)$  dimensions. In fact, according to the Jacobi property of the Poisson brackets we have:

$$\{H(\phi), \{O_I(\phi), O_J(\phi)\}_{pb}\}_{pb} + \{O_I(\phi), \{O_J(\phi), H(\phi)\}_{pb}\}_{pb} + \{O_J(\phi), \{H(\phi), O_I(\phi)\}_{pb}\}_{pb} = 0 \quad (20)$$

Then, since  $O_I(\phi)$  and  $O_J(\phi)$  are constants of the motion in  $\mathcal{D}_{\phi_i}$ , the  $\{O_I(\phi), O_J(\phi)\}_{pb}$  will also be so. As a consequence, if we could define  $N$  constants of the motion such that

$$\{O_I(\phi), O_J(\phi)\}_{pb} = 0 \quad (21)$$

at each point  $\phi \in \mathcal{D}_{\phi_i}^N$  (where  $\mathcal{D}_{\phi_i}^N$  is the already defined domain of  $2N + 1$  dimensions around  $\phi_i$ ) using these functions as initial conditions, we can obtain a complete set of constants of the motion in involution in the domain  $\mathcal{D}_{\phi_i} = \mathcal{D}_{\phi_i}^{N+1}$  of dimension  $2(N + 1)$ , as promised.

Now the problem is reduced to prove the existence of the  $N$   $O_I(\phi)$ ,  $O_J(\phi)$  satisfying eq.(21) in  $\mathcal{D}_{\phi_i}^N$ . Again, the existence of such a set can be easily proved by using the same strategy as above, but now *recursively*. We can begin with an arbitrary function  $O_1(\phi)$  defined in a domain  $\mathcal{D}_{\phi_i}^0$  of  $N + 1$  dimensions. Then, we consider another  $O_2(\phi)$  (defined in a  $N + 2$  dimension domain  $\mathcal{D}_{\phi_i}^1$  containing  $\mathcal{D}_{\phi_i}^0$ ) as the Hamiltonian of eq.(19) and obtain by integration a function  $O_1(\phi)$ , defined in the domain  $\mathcal{D}_{\phi_i}^1$  of  $N + 2$  dimensions, such that in this domain  $\{O_2(\phi), O_1(\phi)\}_{pb} = 0$ . Finally, we iterate the procedure up to find the set of functions in involution in the  $\mathcal{D}_{\phi_i}^N$  of dimensions  $2N + 1$ , which can be taken as initial conditions of eq.(19). In this way, the proof is completed.

c.- Now, in order to go from classical to quantum, we can also extend these local  $O_I(\phi)$ , defined in  $\mathcal{D}_{\phi_i} = \mathcal{D}_{\phi_i}^{N+1}$  of dimensions  $2(N + 1)$ , to all  $\mathcal{M}$  by defining  $O_I(\phi) = 0$  for  $\phi \in \mathcal{M} \setminus \mathcal{D}_{\phi_i}$ . In this case, there will be a jump in the frontier of  $\mathcal{D}_{\phi_i}$ , and the definition will be only continuous a.e. (almost everywhere). Or, on physical grounds, we can take the precaution of joining these zero functions with functions  $O_I(\phi)$  in a zone around  $\mathcal{D}_{\phi_i}$ , that we will call  $\mathcal{F}_{\phi_i}$ , in an smooth way (e.g. by using  $C^r$  functions with an adequate  $r$ ).

Therefore, we have proved the existence of local complete systems of constants of the motion in involution  $\{O_I(\phi)\} = \{H(\phi), O_1(\phi), \dots, O_N(\phi)\}$  that we can extend to all  $\mathcal{M}$ , at least a.e., by adding null functions in  $\mathcal{M} \setminus \mathcal{D}_{\phi_i}^{N+1}$  as explained above. Since they belong to  $\mathcal{D}_{\phi_i}$ , we will call them  $\{H(\phi), O_{\phi_i 1}(\phi), \dots, O_{\phi_i N}(\phi)\}$ . Each system  $\{H(\phi), O_{\phi_i 1}(\phi), \dots, O_{\phi_i N}(\phi)\}$  can be considered as a *local (approximate)  $N + 1$ -CSCO* in  $\mathcal{D}_{\phi_i} = \mathcal{D}_{\phi_i}^{N+1}$  in the sense that, even if it is not an exact CSCO, we can compute their Weyl transformations obtaining

<sup>4</sup>Observe that, if the CSCO has  $< N + 1$  operators, we have not *good* quantum numbers enough to label the eigenvectors.

<sup>5</sup>In the 'old quantization' approach, the problems were certainly more severe.

<sup>6</sup>This fact can be considered as almost evident, but since it is not demonstrated in usual textbooks, we will give a complete demonstration below.

<sup>7</sup>We can also say that the quantum system is *dissymmetrized* [45].

<sup>8</sup>See Appendix A for details. Moreover, a certain determinant  $\Delta$ , defined in [46], must be  $\Delta \neq 0$  in this domain.

$$\{\hat{H}_{\phi_i}, \hat{O}_{\phi_i 1}, \dots, \hat{O}_{\phi_i N}\}$$

and their Wigner transformations are a complete set of constants of the motion in involution in  $\mathcal{D}_{\phi_i}$ . In fact, from eq.(10) we see that

$$\{O_{\phi_i I}(\phi), O_{\phi_j J}(\phi)\}_{mb} = 0(\hbar^2), \quad \text{or} \quad [\hat{O}_{\phi_i I}, \hat{O}_{\phi_j J}] = 0(\hbar^2) \quad (22)$$

namely, they only commute approximately.

Let us now consider in more detail the joining zones  $\mathcal{F}_{\phi_i}$  where we have used  $C^r$ -functions that do not satisfy the required differential equations (19) to (21), in such a way that the terms  $\frac{i\hbar}{2} \overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b$  of eq.(8) produce unwanted contributions of order  $\hbar/PQ$ , where  $P$  and  $Q$  are of the order of magnitude of the jumps in the momentum and configuration variables in the joining zone. Since  $PQ = \varepsilon^2$  is an action measuring the joining zone (where  $\varepsilon$  is the characteristic mean width of the joining zone, precisely  $\varepsilon^{2(N+1)} \cong V_\varepsilon$ , the volume of the joining zones  $\mathcal{F}_{\phi_i}$ ), the unwanted terms are of the order of  $\hbar/\varepsilon^2$ , that is, they are another contribution  $0(\frac{\hbar^2}{\varepsilon^4})$ , or simply  $0(\hbar^2)$ , to add to (22)<sup>9</sup>. Anyhow, these terms will vanish when we make the limit  $\hbar \rightarrow 0$  in Section IV.

Although this approximation seems sufficient for physical purposes, we can even improve it. In fact, we can repeat all the process based on eqs.(19) and (21) substituting them with

$$\{H(\phi), O_I(\phi)\}_{mb} = 0, \quad \{O_I(\phi), O_J(\phi)\}_{mb} = 0 \quad (23)$$

Taking into account eqs.(7) and (10), these are differential equations of infinite order (see Appendix A.c). But we can cut these expansions at any finite order and obtain a system of usual finite differential equations that can be solved; then, we can repeat the procedure described above with any desired precision. In this way, we can eliminate the  $0(\hbar^2)$  coming from eq.(10) but not those coming from the joining zones. Thus, we have again obtained approximate (up to  $0(\hbar^2)$ ) local  $N+1$ -CSCO,<sup>10</sup> and we can define local eigenstates and write equations like (35) for each  $\mathcal{D}_{\phi_i}$ .<sup>11</sup>

d.- Let us observe that natural global coordinates  $\phi = (q, p_p)$  of phase space  $\mathcal{M}$  can be (locally) substituted, by using (local) canonical transformation, with (local) coordinates  $(\theta_{\phi_i I}, O_{\phi_i I})$ , with  $i = 0, 1, \dots, N$  and  $H = O_{\phi_i 0}$ , where the  $\theta_{\phi_i I}(\phi)$  are the coordinates canonically conjugated to the  $O_{\phi_i I}(\phi)$  in  $\mathcal{D}_{\phi_i}$ . The  $(\theta_{\phi_i I}, O_{\phi_i I})$  is clearly a chart of  $\mathcal{M}$  in the domain  $\mathcal{D}_{\phi_i}$ .<sup>12</sup> Since the system is endowed with adequate smooth properties (let us say  $C^r$ ), another similarly constructed chart  $(\theta_{\phi_j I}, O_{\phi_j I})$  in the domain  $\mathcal{D}_{\phi_j}$  is smoothly connected with the previous one at any  $\phi \in \mathcal{D}_{\phi_i} \cap \mathcal{D}_{\phi_j}$  (see demonstration in Section V). Then, the set of all these charts is a  $C^r$ -atlas in  $\mathcal{M}$ . This will be the atlas we will primarily concerned with.

e.- We can also define a (*ad hoc*) positive partition of the identity (see [51] sec. 3.4) in the following sense. Let us define

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<sup>9</sup>Any counterfactual  $0(\hbar)$  is really a factual  $0(\hbar/S)$ , where the action  $S \rightarrow \infty$ . Precisely, this means that  $S \gg S_0$ , where  $S_0$  is a characteristic action. In order to give an example for this  $S_0$ , we can consider the fine structure constant:

$$\alpha = \frac{\hbar}{m_e c a_0} = \frac{\hbar}{S_0} \approx \frac{1}{137}$$

where  $m_e$  is the electron mass,  $c$  is the velocity of light, and  $a_0$  is the Bohr radius. For  $S \gg S_0$  we can neglect  $\alpha$  and, therefore, we loose pure quantum effects, like spin, and pass from the realm of quantum mechanics to the classical limit.

<sup>10</sup>Counterfactually,  $0(\frac{\hbar^2}{\varepsilon^4})$  goes to zero when  $\hbar \rightarrow 0$ . Factually, it goes to zero if the action of the system  $S$  is infinitely large. Then, since  $\varepsilon^2$  has the only constraint  $\varepsilon^2 < S$ , it can be as large as we wish and we have  $\frac{\hbar^2}{\varepsilon^4} \rightarrow 0$  (see also footnote 13).

<sup>11</sup>An example of this phenomenon is the Sinai billiard discussed in Appendix B. Other examples are classical scattering systems: in fact, they have an 'in' CSCO and an 'out' CSCO, which are different since the constants of the motion are not the same in these CSCOs. Another example is the two slits experiment when we mimic the screen with an infinite potential wall: before the screen we have a local CSCO  $\{\hat{H}, \hat{P}\}$ , on the screen the CSCO is  $\{\hat{H}, \hat{X}\}$  since the kinetic term of the Hamiltonian can be neglected with respect to the infinite potential wall, and after the screen again  $\{\hat{H}, \hat{P}\}$ . More complex examples are the so-called pseudointegrable systems ([47], p.98, [48], [49]). Tori become spheres with 'handles' that cannot be covered with a single chart. A further example is Robnik's billiard [50]. Moreover, it is clear that the fractal structure of some examples of chaos breaks the tori completely and, therefore, in this case the radius of the integration domains probably vanish (nevertheless, this would not be a physical case, see Appendix A).

<sup>12</sup>This is not a generic chart, but a very peculiar one, since coordinates  $O_{\phi_i I}$  are constants of the motion satisfying eqs.(19) and (21).

$$1 = I(\phi) = \sum_i B_{\phi_i}(\phi) \quad (24)$$

where  $B_{\phi_i}(\phi)$  are 'bump' functions such that

$$B_{\phi_i}(\phi) \begin{cases} = 1 & \text{if } \phi \in D_{\phi_i} \\ \in [0, 1] & \text{if } \phi \in F_{\phi_i} \\ = 0 & \text{if } \phi \notin D_{\phi_i} \cup F_{\phi_i} \end{cases} \quad (25)$$

where  $D_{\phi_i}$  is a domain and  $F_{\phi_i}$  is the frontier zone around  $D_{\phi_i}$  (the  $F_{\phi_i}$  are similar to the  $\mathcal{F}_{\phi_i}$  but they are related to the  $D_{\phi_i}$ ) defined in such a way that  $D_{\phi_i} \cup F_{\phi_i} \subset \mathcal{D}_{\phi_i}$  and the intersection zones of the  $D$ 's vanish:  $D_{\phi_i} \cap D_{\phi_j} = \emptyset$ . Let us stress that the  $B_{\phi_i}(\phi)$  in the frontier zones satisfy eq.(24). Now, for any  $A(\phi)$  we can define a

$$A_{\phi_i}(\phi) = A(\phi)B_{\phi_i}(\phi)$$

and for any  $A(\phi)$  we have

$$A(\phi) = A(\phi) \sum_i B_{\phi_i}(\phi) = \sum_i A_{\phi_i}(\phi)$$

With the mapping  $\text{sym}b^{-1}$  we find

$$\hat{A} = \text{sym}b^{-1}A(\phi) = \sum_i \text{sym}b^{-1}A_{\phi_i}(\phi) = \sum_i \hat{A}_{\phi_i} \quad (26)$$

where  $\hat{A}_{\phi_i} = \text{sym}b^{-1}A_{\phi_i}(\phi)$  can be considered as a *localization* of  $\hat{A}$  in  $D_{\phi_i}$ . Then, from eq.(26)

$$\hat{A} = \sum_i \hat{A}_{\phi_i} \quad (27)$$

Moreover, since we have a local  $N+1$ -CSCO in each  $D_{\phi_i} \cup F_{\phi_i} \subset \mathcal{D}_{\phi_i}$ , we can decompose

$$\hat{A}_{\phi_i} = \sum_j A_{j\phi_i} |j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} \quad (28)$$

where the  $\{|j\rangle_{\phi_i}^{(A)}\}$  are the corresponding eigenvectors of  $\hat{A}_{\phi_i}$ ; the  $\{\hat{A}_{\phi_i}\}$  can be considered as a local  $N+1$ -CSCO of  $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$ .

Now we can prove that the support of  $\text{sym}b|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)}$  is contained in  $D_{\phi_i} \cup F_{\phi_i}$  i.e. the support of  $\text{sym}b\hat{A}_{\phi_i}$ . In fact, from eq.(28) we have

$$\hat{A}_{\phi_i} |j\rangle_{\phi_i}^{(A)} = A_{j\phi_i} |j\rangle_{\phi_i}^{(A)}$$

or

$$\hat{A}_{\phi_i} |j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} = A_{j\phi_i} |j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)}$$

Then,

$$\text{sym}b\hat{A}_{\phi_i} * \text{sym}b|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} = A_{j\phi_i} \text{sym}b|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} \quad (29)$$

But  $\text{sym}b\hat{A}_{\phi_i}$  and all its derivatives vanish for  $\phi \notin D_{\phi_i} \cup F_{\phi_i}$ . Therefore, if  $A_{j\phi_i} \neq 0$ , this also must happen for  $\text{sym}b|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)}$ , and the support of this function is contained in  $D_{\phi_i} \cup F_{\phi_i}$ . If  $A_{j\phi_i} = 0$ , we can repeat the argument with the operator  $\hat{A}_{\phi_i} + \alpha \hat{B}_{\phi_i}$  and take the limit  $\alpha \rightarrow 0$ , and we will find the same result.

From eq.(28) we have

$$\hat{A} = \sum_{ij} A_{j\phi_i} |j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} \quad (30)$$

all over  $\mathcal{M}$ . Moreover, from eq.(28) we also have

$$\text{symp}\hat{A}_{\phi_i} = \sum_j A_{j\phi_i} \text{symp}|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)}$$

and, as we have just proved,

$$\text{symp}|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)}(\phi) = 0 \text{ if } \phi \notin D_{\phi_i} \subset D_{\phi_i} \cup F_{\phi_i}$$

Then, since for  $i \neq k$ ,  $D_{\phi_i} \cap D_{\phi_k} = \emptyset$  (but  $F_{\phi_i} \cap F_{\phi_j} \neq \emptyset$ ), we have

$$|\langle j|_{\phi_i}^{(A)} |j'\rangle_{\phi_k}^{(A)}|^2 = \langle j|_{\phi_i}^{(A)} |j'\rangle_{\phi_k}^{(A)} \langle j'|_{\phi_k}^{(A)} |j\rangle_{\phi_i}^{(A)} = (|j'\rangle_{\phi_k}^{(A)} \langle j'|_{\phi_k}^{(A)} ||j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)}) =$$

$$\int_{\mathcal{M}} \text{symp}|j'\rangle_{\phi_k}^{(A)} \langle j'|_{\phi_k}^{(A)} \text{symp}|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} d\phi^{2(N+1)} =$$

$$\int_F \text{symp}|j'\rangle_{\phi_k}^{(A)} \langle j'|_{\phi_k}^{(A)} \text{symp}|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} d\phi^{2(N+1)} = 0(\varepsilon^{2(N+1)}) \quad (31)$$

where  $F$  is the union of all the joining zones  $F_{\phi_i}$  and  $\varepsilon$  is the characteristic width of the joining zone. Therefore, for  $i \neq k$  and  $\varepsilon \rightarrow 0$ ,<sup>13</sup> we obtain

$$\langle j|_{\phi_i}^{(A)} |j'\rangle_{\phi_k}^{(A)} = 0 \quad (32)$$

This means that, in the limit  $\varepsilon \rightarrow 0$ , decomposition (30) is an *orthogonal decomposition* in the  $|j\rangle_{\phi_i}^{(A)}$ .

**Remark.** Let us now discuss the subspaces that can be defined by the above decomposition. Being  $\{|j\rangle_{\phi_i}^{(A)}\}$  a basis of the Hilbert space  $\mathcal{H}$  where we are working, any  $|\varphi\rangle \in \mathcal{H}$  could be decomposed as

$$|\varphi\rangle = \sum_{ij} \varphi_{ij} |j\rangle_{\phi_i}^{(A)} = \sum_i |\varphi_i\rangle$$

where

$$|\varphi_i\rangle = \sum_j \varphi_{ij} |j\rangle_{\phi_i}^{(A)}$$

and this ket belongs to subspace  $\mathcal{H}_i$  of  $\mathcal{H}$ . Then,

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i \quad (33)$$

Nevertheless, from eq.(28) where the off-diagonal terms  $i \neq j$  are absent, we have

<sup>13</sup>Precisely: let us call  $V_{\mathcal{M}}$  the volume of phase space:  $V_{\mathcal{M}} \sim S^{N+1}$ . Analogously,

$$I_{\mathcal{M}} = \int_{\mathcal{M}} \text{symp}|j'\rangle_{\phi_k}^{(A)} \langle j'|_{\phi_k}^{(A)} \text{symp}|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} d\phi^{2(N+1)} \sim V_{\mathcal{M}} \sim S^{N+1}$$

Let us also define  $I_{\varepsilon} = \int_{\mathcal{F}} \text{symp}|j'\rangle_{\phi_k}^{(A)} \langle j'|_{\phi_k}^{(A)} \text{symp}|j\rangle_{\phi_i}^{(A)} \langle j|_{\phi_i}^{(A)} d\phi^{2(N+1)} \sim V_{\varepsilon} = \varepsilon^{2(N+1)}$ . In order to prove eq.(31), it is necessary that  $I_{\varepsilon} \ll I_{\mathcal{M}}$  in such a way that  $I_{\varepsilon}$  could be neglected. But  $I_{\varepsilon} \sim V_{\varepsilon}$  and  $I_{\mathcal{M}} \sim S^{N+1}$ ; then,  $\varepsilon^2 \ll S$ .

Therefore,  $\varepsilon$  must be:

- 1.- Such that the ratio  $\frac{\hbar}{\varepsilon^2}$  be negligible to eliminate the unwanted terms  $\frac{i\hbar}{2} \overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b$  in the joining zone (see footnote 10).
- 2.- As small as  $\varepsilon^2 \ll S$  to satisfy eq.(31).

Since  $\hbar \ll S$ , we can satisfy both conditions with an adequate  $\varepsilon$ , namely, such that:

$$\hbar \ll \varepsilon^2 \ll S$$



$$\widehat{A}_{\phi_i} \in \mathcal{H}_i \otimes \mathcal{H}_i = \mathcal{O}_i$$

and from  $\widehat{A} \in \mathcal{H} \otimes \mathcal{H}$  and eq.(27) we have<sup>14</sup>

$$\mathcal{O} = \mathcal{H} \otimes \mathcal{H} = \bigoplus_i \mathcal{O}_i = \bigoplus_i \mathcal{H}_i \otimes \mathcal{H}_i \quad (34)$$

This shows that *there are no cross terms*  $\mathcal{H}_i \otimes \mathcal{H}_j$  in the decomposition of  $\mathcal{O} = \mathcal{H} \otimes \mathcal{H}$ .

We can see that the decomposition that really matters for our discussion is (34), the decomposition in subspaces  $\mathcal{O}_i$ , and not (33): the repeated index  $i$  in the basis  $\{|j\rangle_{\phi_i}^{(A)} \langle j'|_{\phi_i}^{(A)}\}$  means that this basis corresponds to the decomposition done in the  $\mathcal{O}_i$ , which is the relevant one for this paper.

## B. Decoherence in the energy

a.- We will now introduce decoherence according to the self-induced approach. Let us define, *in each*  $D_{\phi_i}$ , a local  $N + 1$ -CSCO where, as in eq.(30), the observables of the  $N + 1$ -CSCO  $\{\widehat{H}, \widehat{O}_{\phi_i}\}$  are decomposed as

$$\widehat{H} = \int_0^\infty \omega \sum_{im} |\omega, m\rangle_{\phi_i} \langle \omega, m|_{\phi_i} d\omega, \quad \widehat{O}_{\phi_i I} = \int_0^\infty \sum_m O_{mI\phi_i} |\omega, m\rangle_{\phi_i} \langle \omega, m|_{\phi_i} d\omega \quad (35)$$

where the energy spectrum is  $0 \leq \omega < \infty$  and  $m_{I\phi_i} = \{m_{1\phi_i}, \dots, m_{N\phi_i}\}$ ,  $m_{I\phi_i} \in \mathbb{N}$  (the spectra of the  $\widehat{O}_{\phi_i I}$  are discrete for simplicity).<sup>15</sup> Therefore

$$\widehat{H}|\omega, m\rangle_{\phi_i} = \omega|\omega, m\rangle_{\phi_i}, \quad \widehat{O}_{\phi_i I}|\omega, m\rangle_{\phi_i} = O_{mI\phi_i}|\omega, m\rangle_{\phi_i} \quad (36)$$

where the  $|\omega, m\rangle_{\phi_i}$  are the eigenvectors of the observables  $\widehat{H}$ , and  $\widehat{O}_{\phi_i}$  (such that  $\text{symp}\widehat{O}_{\phi_i} = O_{\phi_i}(\phi) \neq 0$  only in  $D_{\phi_i} \cap F_{\phi_i}$ ) and  $m$  is a shorthand for  $m_{\phi_i I} = \{m_{\phi_i 1}, \dots, m_{\phi_i N}\}$ . The set  $\{|\omega, m\rangle_{\phi_i}\}$  is orthonormal in  $\omega$  and in  $m$ , in the usual eigenvalue indices and in  $i$ , as proved in eq.(32):

$$\langle \omega, m|_{\phi_i} | \omega', m'\rangle_{\phi_j} = \delta(\omega - \omega') \delta_{mm'} \delta_{ij} \quad (37)$$

b.- Now we can define our relevant algebra of observables. This choice *will play the role of coarse-graining* in our approach. A generic observable reads, in the orthonormal basis just defined,

$$\widehat{O} = \sum_{imm'} \int_0^\infty \int_0^\infty d\omega d\omega' \widetilde{O}(\omega, \omega')_{\phi_i mm'} |\omega, m\rangle_{\phi_i} \langle \omega', m'|_{\phi_i} \quad (38)$$

where  $\widetilde{O}(\omega, \omega')_{\phi_i mm'}$  is a generic kernel or distribution in  $\omega, \omega'$ .<sup>16</sup> But we must restrict this set of observables since it is too large for our purposes; furthermore, it is not easy to work with generic kernels or distributions. However, we cannot make the algebra too small either. In fact, let us suppose that, in order to make computation easier, we postulate that the  $\widetilde{O}(\omega, \omega')_{\phi_i mm'}$  be just regular functions. Then, the states read

$$\widehat{\rho} = \sum_{imm'} \int_0^\infty \int_0^\infty d\omega d\omega' \widetilde{\rho}(\omega, \omega')_{\phi_i mm'} |\omega, m\rangle_{\phi_i} \langle \omega', m'|_{\phi_i}$$

where the  $\widetilde{\rho}(\omega, \omega')_{\phi_i mm'}$ , in the dual space, are also regular functions. Then,

<sup>14</sup>This decomposition is similar to the decomposition of a function in its even and odd parts in a Fourier transformation, where the sin is the even basis and the cos is the odd basis.

<sup>15</sup>Hamiltonians with continuous spectra are considered in papers [23] and [24]. We use this kind of spectra since they are the usual ones in the macroscopic limit  $\hbar \rightarrow 0$  (see [47] eq.(3.1.24) p.67). Strictly, we should call  $|\omega, m\rangle_{\phi_i}^{(\widehat{H}, \widehat{O}_{\phi_i})}$  the vectors  $|\omega, m\rangle_{\phi_i}$ , but we will just call them  $|\omega, m\rangle_{\phi_i}$  for simplicity.

<sup>16</sup>As explained at the end of the last subsection, the index  $i$  in projector  $|\omega, m\rangle_{\phi_i} \langle \omega', m'|_{\phi_i}$  corresponds to the fact that the decomposition is done in the  $\mathcal{O}_i$  and, therefore, the index is repeated in  $|\omega, m\rangle_{\phi_i}$  and in  $\langle \omega', m'|_{\phi_i}$ .

$$\langle \widehat{O} \rangle_{\widehat{\rho}(t)} = \sum_{imm'} \int_0^\infty \int_0^\infty d\omega d\omega' \overline{\widehat{\rho}(\omega, \omega')_{\phi_i mm'}} e^{i(\omega - \omega')t} \widetilde{O}(\omega, \omega')_{\phi_i mm'}$$

and, since the product  $\overline{\widehat{\rho}(\omega, \omega')_{\phi_i mm'}} \widetilde{O}(\omega, \omega')_{\phi_i mm'}$  is a regular function (i.e.  $\mathbb{L}_1$  in  $\nu = \omega - \omega'$ ), as a result of the Riemann-Lebesgue theorem the mean value  $\langle \widehat{O} \rangle_{\widehat{\rho}(t)}$  would vanish for  $t \rightarrow \infty$ : we would obtain destructive interference not only for the off-diagonal terms, *but for all of them*. On the contrary, if  $\overline{\widehat{\rho}(\omega, \omega')_{\phi_i mm'}}$  and  $\widetilde{O}(\omega, \omega')_{\phi_i mm'}$  were generic kernels, we could not use the Riemann-Lebesgue theorem, and we can presume that there will be no destructive interference. This means that  $\widetilde{O}(\omega, \omega')_{\phi_i mm'}$  cannot be *so regular* nor *so non-regular*: we must choose something in between. In order to avoid these unacceptable results, the simplest choice is the van Hove choice; so, as in paper [26], we will take:

$$\widetilde{O}(\omega, \omega')_{\phi_i mm'} = O(\omega)_{\phi_i mm'} \delta(\omega - \omega') + O(\omega, \omega')_{\phi_i mm'} \quad (39)$$

where the  $O(\omega, \omega')_{\phi_i mm'}$  are ordinary functions of the real variables  $\omega$  and  $\omega'$  (these functions must have some mathematical properties in order to develop the theory; these properties are listed in paper [24]). This choice is theoretically explained in papers [36], [37], [38], [39], [40], [41], and [26]. Moreover, we need the  $\delta(\omega - \omega')$  term in order that the members of the  $N + 1$ -CSCO of eq.(35) be contained in the space of observables. So our operator belongs to an algebra  $\widehat{\mathcal{A}}$  (defined by eq.(39) and the properties just required for the  $O(\omega, \omega')_{\phi_i mm'}$ ), and reads

$$\widehat{O} = \sum_{imm'} \int_0^\infty d\omega O(\omega)_{\phi_i mm'} |\omega, m\rangle_{\phi_i} \langle \omega, m'|_{\phi_i} + \sum_{imm'} \int_0^\infty \int_0^\infty d\omega d\omega' O(\omega, \omega')_{\phi_i mm'} |\omega, m\rangle_{\phi_i} \langle \omega', m'|_{\phi_i} \quad (40)$$

The first term in the r.h.s. will be called  $\widehat{O}_S$ , the *singular* component, and the second term will be called  $\widehat{O}_R$ , the *regular* component,<sup>17</sup> and  $[\widehat{H}, \widehat{O}_S] = 0$ . The *observables* are the self-adjoint  $O^\dagger = O$  operators. We will say that these observables belong to a space  $\widehat{\mathcal{O}}$  (which is contained in the operator algebra  $\widehat{\mathcal{A}}$ );  $\{|\omega, m, m'\rangle_{\phi_i}, |\omega, \omega', m, m'\rangle_{\phi_i}\}$  is a basis of this space, where

$$|\omega, m, m'\rangle_{\phi_i} \doteq |\omega, m\rangle_{\phi_i} \langle \omega, m'|_{\phi_i}, \quad |\omega, \omega', m, m'\rangle_{\phi_i} \doteq |\omega, m\rangle_{\phi_i} \langle \omega', m'|_{\phi_i} \quad (41)$$

Then, the classical analogue of eq.(40) would be

$$O(\phi) = \sum_{imm'} \int_0^\infty d\omega O(\omega)_{\phi_i mm'} |\omega, m, m'(\phi)\rangle_{\phi_i} + \sum_{imm'} \int_0^\infty \int_0^\infty d\omega d\omega' O(\omega, \omega')_{\phi_i mm'} |\omega, \omega', m, m'(\phi)\rangle_{\phi_i}$$

where  $|\omega, m, m'(\phi)\rangle_{\phi_i} = \text{sympb}|\omega, m, m'\rangle_{\phi_i}$  and  $|\omega, \omega', m, m'(\phi)\rangle_{\phi_i} = \text{sympb}|\omega, \omega', m, m'\rangle_{\phi_i}$ .

c.- The quantum states  $\widehat{\rho}$  are measured by the observables just defined, leading to the mean values of these observables; in the usual notation:  $\langle \widehat{O} \rangle_{\widehat{\rho}} = \text{Tr}(\widehat{\rho}^\dagger \widehat{O})$ . We can conceive that mean values as the more primitive objects of the quantum theory (see [52]). These mean values, generalized as in paper [24] and symbolized as  $(\widehat{\rho}|\widehat{O})$ , can be considered as the result of the action of the linear functionals  $\widehat{\rho}$  on the observables of the vector space  $\widehat{\mathcal{O}}$ . Then,  $\widehat{\rho} \in \widehat{\mathcal{S}} \subset \widehat{\mathcal{O}}'$ , where  $\widehat{\mathcal{S}}$  is a convenient (i.e. satisfying eqs.(45) and (46) below) convex set contained in  $\widehat{\mathcal{O}}'$ , the space of linear functionals over  $\widehat{\mathcal{O}}$ . The basis of  $\widehat{\mathcal{O}}'$  (that is, the *co-basis* of  $\widehat{\mathcal{O}}$  in each  $D_{\phi_i}$ ) is  $\{(\omega, mm'|_{\phi_i}, (\omega\omega', mm'|_{\phi_i})\}$ , and it is defined in terms of its functionals by the equations

$$\begin{aligned} (\omega, m, m'|_{\phi_i} | \eta, n, n')_{\phi_j} &= \delta(\omega - \eta) \delta_{mn} \delta_{m'n'} \delta_{ij} \\ (\omega, \omega', m, m'|_{\phi_i} | \eta, \eta', n, n')_{\phi_j} &= \delta(\omega - \eta) \delta(\omega' - \eta') \delta_{mn} \delta_{m'n'} \delta_{ij} \end{aligned} \quad (42)$$

and all other  $(\cdot|\cdot)$  are zero. The orthogonality in  $i, j, \dots$  is a consequence of eqs.(37) and (41). Let us observe that  $(\omega, \omega', m, m'|_{\phi_i} \doteq |\omega, m\rangle_{\phi_i} \langle \omega', m'|_{\phi_i}$  but  $(\omega, m, m'|_{\phi_i} \neq |\omega, m\rangle_{\phi_i} \langle \omega, m'|_{\phi_i}$ .<sup>18</sup> Then, a generic quantum state reads

<sup>17</sup>The component  $\widehat{O}_S$  is called singular because it contains a hidden distribution  $\delta(\omega - \omega')$ . In fact, it can be obtained from the regular part by making  $O(\omega, \omega')_{\phi_i mm'} = O(\omega)_{\phi_i mm'} \delta(\omega - \omega')$ .

<sup>18</sup>If  $(\omega, m, m'|_{\phi_i} = |\omega, m\rangle_{\phi_i} \langle \omega, m'|_{\phi_i}$ , it is easy to show that a divergence appears.

$$\hat{\rho} = \sum_{imm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{\phi_i mm'} (\omega, mm'|_{\phi_i} + \sum_{imm'} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho(\omega, \omega')}_{\phi_i mm'} (\omega\omega', mm'|_{\phi_i} \quad (43)$$

This is also the case for the corresponding classical analogue  $\rho(\phi)$  of  $\hat{\rho}$ ,

$$\rho(\phi) = \sum_{imm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{\phi_i mm'} (\omega, mm'(\phi)|_{\phi_i} + \sum_{imm'} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho(\omega, \omega')}_{\phi_i mm'} (\omega\omega', mm'(\phi)|_{\phi_i} \quad (44)$$

Each of the terms of the sum  $\sum_i$  can be considered as a term of a decomposition, where each of the  $\rho_{\phi_i}(\phi, t) = \text{symb}\hat{\rho}_{\phi_i}(t)$  does not vanish in the corresponding domain  $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$ .<sup>19</sup> As before, the first term in the r.h.s. of eq.(43) will be called  $\hat{\rho}_S$ , the *singular* component, and the second term will be called  $\hat{\rho}_R$ , the *regular* component. Functions  $\rho(\omega, \omega')_{\phi_i mm'}$  are regular (see [24] for details).

Going back to the decomposition (44), we impose the following conditions. We require that  $\hat{\rho}^\dagger = \hat{\rho}$ , i.e.

$$\overline{\rho(\omega, \omega')}_{\phi_i mm'} = \rho(\omega', \omega)_{\phi_i m' m} \quad (45)$$

and that  $\overline{\rho(\omega)}_{\phi_i mm'}$  be *real and non-negative*, satisfying the total probability condition

$$(\hat{\rho}|\hat{I}) = \sum_{im} \int_0^\infty d\omega \rho(\omega)_{\phi_i} = 1 \quad (46)$$

where  $\hat{I} = \int_0^\infty d\omega \sum_{im} |\omega, m\rangle_{\phi_i} \langle \omega, m|_{\phi_i}$  is the identity operator (24) in  $\hat{\mathcal{O}}$  represented in each  $D_{\phi_i}$ . Eq.(46) is the extension to state functionals of the usual condition  $\text{Tr}\rho^\dagger = 1$ , when  $\rho$  is a density operator. Thus, from now on,  $\text{Tr}\rho \doteq (\rho|I)$ . For these reasons,  $\hat{\rho}$  belongs to the already defined convex set  $\hat{\mathcal{S}} \subset \hat{\mathcal{O}}'$ . The time evolution of the quantum state  $\hat{\rho}$  reads

$$\hat{\rho}(t) = \sum_{imm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{\phi_i mm'} (\omega, mm'|_{\phi_i} + \sum_{imm'} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho(\omega, \omega')}_{\phi_i mm'} e^{i(\omega - \omega')t/\hbar} (\omega\omega', mm'|_{\phi_i} \quad (47)$$

As we have already said, at the statistical quantum level we can only measure mean values of observables in quantum states

$$\langle \hat{O} \rangle_{\hat{\rho}(t)} = (\hat{\rho}(t)|\hat{O}) = \sum_{imm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{\phi_i mm'} O(\omega)_{\phi_i mm'} + \sum_{imm'} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho(\omega, \omega')}_{\phi_i mm'} e^{i(\omega - \omega')t/\hbar} O(\omega, \omega')_{\phi_i mm'} \quad (48)$$

From eq.(16), the classical analogue has exactly the same form

$$\begin{aligned} \langle O(\phi) \rangle_{\rho(\phi, t)} &= (\rho(\phi, t)|O(\phi)) = \sum_{imm'} \int_0^\infty d\omega \overline{\rho(\omega)}_{\phi_i mm'} O(\omega)_{\phi_i mm'} + \\ &+ \sum_{imm'} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho(\omega, \omega')}_{\phi_i mm'} e^{i(\omega - \omega')t/\hbar} O(\omega, \omega')_{\phi_i mm'} \end{aligned} \quad (49)$$

Both decompositions are valid in each  $D_{\phi_i}$ . If we take into account that  $O(\omega, \omega')$  and  $\overline{\rho(\omega, \omega')}_{\phi_i mm'}$  are regular (as regular as needed to use the Riemann-Lebesgue theorem, i. e.  $O(\omega, \omega') \overline{\rho(\omega, \omega')}_{\phi_i mm'} \in \mathbb{L}_1(\omega - \omega')$ , see [24]), we can

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<sup>19</sup>Considering for a moment the larger domains  $\mathcal{D}_{\phi_i}$ , of course there are other charts defined in other domains  $\mathcal{D}'_{\phi_k}$  and, therefore, other  $D'_{\phi_k} \subset \mathcal{D}'_{\phi_k}$ . But since  $\rho(\phi, t) = \text{symb}\hat{\rho}(t)$  is defined in the whole  $\mathcal{M}$ , the  $\mathcal{D}_{\phi_i}$ ,  $\mathcal{D}'_{\phi_k}$  are just *local* charts for which the same function  $\rho(\phi, t) = \text{symb}\hat{\rho}(t)$  is *globally* defined in phase space. Moreover, at  $\phi \in \mathcal{D}_{\phi_i} \cap \mathcal{D}_{\phi_j}$ , any pair of charts can be  $C^r$ -smoothly connected in the sense that all their elements can be smoothly connected among each other. The same argument can be applied to the partial decompositions of  $\rho_R(\phi, t) = \text{symb}\hat{\rho}_R(t)$ ,  $\rho_S(\phi, t) = \text{symb}\hat{\rho}_S(t)$  (i.e. the first and second terms of the r.h.s. of the last equation), since both regular and singular parts are also *globally* defined in phase space.

take the limit  $t \rightarrow \infty$  and use the Riemann-Lebesgue theorem. As the result, we see that the fluctuating-regular part vanishes and we arrive to the weak (quantum and classical) limits

$$\begin{aligned} W \lim_{t \rightarrow \infty} \hat{\rho}(t) &= \widehat{\rho}_S = \hat{\rho}_* = \sum_{imm'} \int_0^\infty d\omega \overline{\rho(\omega, p)}_{\phi_i mm'}(\omega, m, m')|_{\phi_i} \\ W \lim_{t \rightarrow \infty} \rho(\phi, t) &= \rho_S(\phi) = \rho_*(\phi) = \sum_{imm'} \int_0^\infty d\omega \overline{\rho(\omega, p)}_{\phi_i mm'}(\omega, m, m', (\phi)|_{\phi_i}) \end{aligned} \quad (50)$$

where  $(\omega, m, m', (\phi)|_{\phi_i}) = \text{symp}(\omega, mm'|_{\phi_i})$  and  $\rho_*(\phi) = \text{symp}\hat{\rho}_*$  are defined in  $\mathcal{M}$  and the integral in eq.(50) is decomposed in different ways at each  $D_{\phi_i}$ . Since only the singular diagonal terms remain, we have obtained decoherence in the energy variable  $\omega$ . Precisely, any quantum state weakly tends to a linear combination of the energy diagonal states  $(\omega, m, m')|_{\phi_i}$  (the energy 'off-diagonal' states  $(\omega, \omega', m, m')|_{\phi_i}$  are not present in  $\rho_*$ ). This is the case if we observe and measure the system evolution with *any possible observable of space*  $\hat{O}$ . Therefore, from the observational point of view, we have decoherence of the energy levels in spite of the fact that, from the strong limit point of view, the off-diagonal terms *never vanish*: they just oscillate since we cannot directly use the Riemann-Lebesgue theorem in the operator equation (47).

#### Important remarks

i.- It may be supposed that decoherence takes place without a coarse-graining. It is not so: the choice of the algebra  $\hat{\mathcal{A}}$  among all possible algebras (see under eq.(39)) and the systematic use of mean values  $\langle \hat{O} \rangle_{\hat{\rho}(t)} = (\hat{\rho}(t)|\hat{O})$  (eq.(48)), restrict the available information and produce the effect of a coarse-graining. In fact, we can define the projector  $\Pi = |\hat{O}\rangle\langle \hat{O}|$ , with  $|\hat{O}\rangle \in \hat{\mathcal{A}}$  and  $\langle \hat{O} | \hat{O} \rangle = 1$ , that projects  $|\hat{\rho}(t)\rangle$  as  $|\hat{\rho}(t)\rangle \Pi = \langle \hat{O} | \hat{\rho}(t) \rangle |\hat{O}\rangle$ , and translates everything in projectors language: we obtain, from eq.(50),  $\lim_{t \rightarrow \infty} (\hat{\rho}(t)|\Pi) = (\hat{\rho}_*|\Pi)$ . This projection will obviously break the unitarity of the primitive evolution. In this way we could develop a formalism closer to the usual one. See a detailed explanation in [13] and [34].

ii.- Theoretically, decoherence takes place at  $t \rightarrow \infty$ . But, in practice, decoherence appears at a decoherence time, as we have defined in [53]: the decoherence time can be easily computed from the poles of the resolvent or the initial conditions density in the complex extension of the  $\hat{H}$  spectrum. Trivial  $\hat{H}$  (e.g. free particle  $\hat{H}$ ) and trivial initial conditions (e.g. zero temperature ones) do not have poles and the decoherence time is infinite. This means that, to reach equilibrium in a finite characteristic time,  $\hat{H}$  must be non-trivial (e.g. the sum of a free Hamiltonian plus an interaction Hamiltonian) and/or the initial conditions must be non-trivial (e.g.  $T \neq 0$ ). For details, see [54], where decoherence times are estimated in  $10^{-13} - 10^{-15} s$  for microscopic bodies, and  $10^{-37} - 10^{-39} s$  for macroscopic bodies; for a thermal bath our results coincide with those obtained by the einselection approach.

### C. Decoherence in the remaining variables

Having obtained decoherence in the energy levels, we must consider decoherence in the other dynamical variables  $O_{\phi_i I}$  of the set of local CSCOs we are using. We will call these variables 'momentum variables'. Since the expression of  $\rho_*$ , given in eq.(50), only involves the time independent components of  $\rho(t)$ , it is impossible that a further decoherence process eliminates the off-diagonal terms in the remaining  $N$  dynamical momentum variables. Therefore, the only alternative is to find the basis where these off-diagonal components  $\rho(\omega)_{\phi_i mm'}$  vanish at any time.

Let us consider the following unitary change of basis

$$|\omega, p\rangle_{\phi_i} = \sum_m U(\omega)_{mp} |\omega, m\rangle_{\phi_i} \quad (51)$$

where  $p$  and  $m$  are shorthand notations for  $p \doteq \{p_1, \dots, p_N\}$  and  $m \doteq \{m_1, \dots, m_N\}$ , and  $[U(\omega)^{-1}]_{mp} = \overline{U(\omega)_{pm}}$ . We choose the new basis  $\{|\omega, p\rangle_{\phi_i}\}$  such that it verifies the generalized orthogonality condition

$$\langle \omega, p |_{\phi_i} | \omega', p' \rangle_{\phi_i} = \delta(\omega - \omega') \delta_{pp'}$$

Since  $\overline{\rho(\omega)}_{\phi_i} = \rho(\omega)_{\phi_i}$ , it is possible to choose  $U(\omega)$  in such a way that the off-diagonal parts of  $\rho(\omega)_{\phi_i pp'}$  vanish, i.e.

$$\rho(\omega)_{\phi_i pp'} = \rho_{\phi_i}(\omega)_p \delta_{pp'} \quad (52)$$

This means that there is a *final local pointer basis* in  $D_{\phi_i}$  for the observables, given by  $\{|\omega, p, p'\rangle_{\phi_i}, |\omega, \omega', p, p'\rangle_{\phi_i}\}$  and defined as in eq.(41) but now with the  $p$ . The corresponding final pointer basis for the states,  $\{(\omega, p, p')|_{\phi_i}, (\omega, \omega', p, p')|_{\phi_i}\}$ , diagonalizes the time independent part of  $\rho(t)$  and, therefore, diagonalizes the final state  $\rho_*$

Now, we have diagonalized the  $\overline{\rho(\omega)}_{\phi_i mm'}$  in  $m$  and  $m'$ , obtaining

$$W \lim_{t \rightarrow \infty} \hat{\rho}(t) = \hat{\rho}_S = \hat{\rho}_* = \sum_{ip} \int_0^\infty d\omega \overline{\rho_{\phi_i}(\omega)}_p (\omega, p, p|_{\phi_i}$$

$$W \lim_{t \rightarrow \infty} \rho(\phi, t) = \rho_S(\phi) = \rho_*(\phi) = \sum_{ip} \int_0^\infty d\omega \overline{\rho_{\phi_i}(\omega)}_p (\omega, p, p, (\phi)|_{\phi_i} \quad (53)$$

Here we are using a local pointer  $N+1$ -CSCO  $\{\hat{H}, \hat{P}_{\phi_i 1}, \dots, \hat{P}_{\phi_i N}\}$  at each  $D_{\phi_i}$ , where the  $\hat{P}_{\phi_i I}$  are

$$\hat{P}_{\phi_i I} = \sum_i \int_0^\infty d\omega \sum_p p_{\phi_i I}(\omega) |\omega, p, p\rangle_{\phi_i} \quad (54)$$

and their classical analogues

$$P(\phi)_{\phi_i I} = \sum_i \int_0^\infty d\omega \sum_p p_{\phi_i I}(\omega) |\omega, p, p(\phi)\rangle_{\phi_i} \quad (55)$$

where  $|\omega, p, p\rangle_{\phi_i} = |\omega, p\rangle_{\phi_i} \langle\omega, p|_{\phi_i}$  or simply  $\{|\omega, p\rangle_{\phi_i}\}$  is the *local pointer basis* in  $D_{\phi_i}$ ; so, we can write eq.(40) in this new basis (see eq.(58) below).<sup>20</sup> Now all the operators and matrices involved are diagonal, and decoherence is complete. We can define all the observables  $\hat{O}$  of eq.(40) in this new local pointer basis.

Since in the limit  $\hbar \rightarrow 0$  we usually have  $\hat{P}$  with continuous spectra, instead of the last equations we would have the natural analogues of eqs.(53) (see [31] and [33] for details)

$$W \lim_{t \rightarrow \infty} \hat{\rho}(t) = \hat{\rho}_S = \hat{\rho}_* = \sum_i \int_0^\infty d\omega \int_{p \in D_{\phi_i}} dp^N \overline{\rho(\omega)_{\phi_i}} (\omega, p, p|_{\phi_i}$$

$$W \lim_{t \rightarrow \infty} \rho(\phi, t) = \rho_S(\phi) = \rho_*(\phi) = \sum_i \int_0^\infty d\omega \int_{p \in D_{\phi_i}} dp^N \overline{\rho(\omega)_{\phi_i}} (\omega, p, p, (\phi)|_{\phi_i} \quad (56)$$

In the next section we will consider the classical limit and, then, we will only use continuous spectra and equations like the last two:<sup>21</sup> so we will re-write some equations in the new basis for the sake of completeness.

<sup>20</sup>The complexity of these formulae demonstrates why it was so difficult to define the pointer basis in a general case. As we can see, the pointer basis depends on  $H$  and the initial conditions, but there are some cases (see section IV) where it only depends on  $H$ .

<sup>21</sup>If we use the Heisenberg picture, the  $\hat{A}$  would become diagonal. So, heuristically

$$\lim_{t \rightarrow \infty} (\hat{\rho}_* | [\hat{A}(t), \hat{B}]) = \lim_{t \rightarrow \infty} Tr(\hat{\rho}_* \hat{A}(t) \hat{B} - \hat{\rho}_* \hat{B} \hat{A}(t)) =$$

$$Tr(\hat{\rho}_* \hat{A}_* \hat{B} - \hat{\rho}_* \hat{B} \hat{A}_*) = Tr(\hat{\rho}_* \hat{A}_* \hat{B} - \hat{A}_* \hat{\rho}_* \hat{B}) = 0$$

where  $\hat{A}_*$  is the diagonal weak limit of  $\hat{A}$  and, therefore, commutes with the diagonal  $\hat{\rho}_*$ . As a consequence, the evolution is (*heuristically*) *weakly asymptotically abelian* ([51], Def. 4.11) since, in the limit  $t \rightarrow \infty$ ,  $\hat{A}$  can be considered commutative. Therefore, a quantum system with continuous spectrum is *weakly asymptotically abelian*.

## IV. THE CLASSICAL STATISTICAL LIMIT

### A. Quantum and classical operators

a.- From now on, we will consider a system from the point of view of the local pointer complete set of  $N + 1$ -commuting observables  $\{\widehat{H}, \widehat{P}_{\phi_1}, \dots, \widehat{P}_{\phi_N}\}$ , defined in eqs.(35) and (54). As above, to simplify the notation we will just call  $\{\widehat{H}, \widehat{P}_{\phi_i}\}$  the set  $\{\widehat{H}, \widehat{P}_{\phi_1}, \dots, \widehat{P}_{\phi_N}\}$ . Thus, we will consider the orthonormal eigenbasis  $\{|\omega, p\rangle_{\phi_i}\}$  of  $\{\widehat{H}, \widehat{P}_{\phi_i}\}$ ,<sup>22</sup> and write the Hamiltonian and  $\widehat{P}$  as

$$\widehat{H} = \sum_i \int_{p \in D_{\phi_i}} dp^N \int_0^\infty \omega |\omega, p\rangle_{\phi_i} \langle \omega, p|_{\phi_i} d\omega \quad \widehat{P}_{\phi_i} = \int_{p \in D_{\phi_i}} dp^N \int_0^\infty p |\omega, p\rangle_{\phi_i} \langle \omega, p|_{\phi_i} d\omega \quad (57)$$

Furthermore, we will consider the algebra  $\widehat{\mathcal{A}}$  of the operators (40), which now read

$$\begin{aligned} \widehat{O} = & \sum_i \int_{p \in D_{\phi_i}} dp^N \int_0^\infty O_{\phi_i}(\omega, p) |\omega, p\rangle_{\phi_i} \langle \omega, p|_{\phi_i} d\omega \\ & + \sum_i \int_{p \in D_{\phi_i}} \int_{p' \in D_{\phi_i}} dp^N dp'^N \int_0^\infty \int_0^\infty O_{\phi_i}(\omega, \omega', p, p') |\omega, p\rangle_{\phi_i} \langle \omega, p'|_{\phi_i} d\omega d\omega' \end{aligned} \quad (58)$$

As before, the first term in the r.h.s. will be called  $\widehat{O}_S$ , the *singular* component, and the second term will be called  $\widehat{O}_R$ , the *regular* component. Also as before, functions  $O_{\phi_i}(\omega, \omega', p, p')$  are regular (see [24] for details),  $[\widehat{H}, \widehat{O}_S] = 0$ ,  $\widehat{O}_S \in \widehat{\mathcal{L}}_S$ , where  $\widehat{\mathcal{L}}_S$  is the singular space,  $\widehat{O}_R \in \widehat{\mathcal{L}}_R$ , where  $\widehat{\mathcal{L}}_R$  is the regular space, and  $\widehat{\mathcal{A}} = \widehat{\mathcal{L}}_S \oplus \widehat{\mathcal{L}}_R$ . The observables are the self-adjoint operators of  $\widehat{\mathcal{A}}$ , and they belong to a space  $\widehat{\mathcal{O}}$ .

b.- Let us now consider the Wigner transformation of these objects. The operators of  $\widehat{\mathcal{L}}_R$  are regular; so, their transformation is obtained as explained in Section II. Then, we have to consider only the singular space  $\widehat{\mathcal{L}}_S$ , the space of the operators that commute with  $\widehat{H}$ . This is not a regular space of operators on a Hilbert space  $\mathcal{H}$  as  $\widehat{\mathcal{L}}_R$ , since it contains a hidden  $\delta(\omega - \omega')$  (see eq.(39)), but the mapping *symp* given by eq.(4) can also be well defined for the observables in  $\widehat{\mathcal{L}}_S$ . In fact, from eq.(58) we know that

$$\widehat{O}_S = \sum_i \int_{p \in D_{\phi_i}} dp^N \int_0^\infty O_{\phi_i}(\omega, p) |\omega, p\rangle_{\phi_i} \langle \omega, p|_{\phi_i} d\omega \quad (59)$$

If we consider, as usual, first  $O_{\phi_i}$  as a polynomial, and then  $O_{\phi_i}$  as a function of a certain space where the polynomials are dense,<sup>23</sup> by using eqs.(57) we can conclude that

$$\widehat{O}_S = \sum_i O_{\phi_i}(\widehat{H}, \widehat{P}_{\phi_i}) = \sum_i \widehat{O}_{S\phi_i} \quad (60)$$

But, when  $\widehat{f}, \widehat{g}$  commute, as the members of the CSCO do (see eq.(12)),

$$\text{symp}(\widehat{f}, \widehat{g}) = (f * g)(\phi) = f(\phi)g(\phi) + 0(\hbar^2) \quad (61)$$

Then, by means of the same procedure as before and eq.(7),

$$\text{symp}\widehat{O}_S = O_S(\phi) = \sum_i O_{\phi_i}(H(\phi), P_{\phi_i}(\phi)) + 0(\hbar^2) = \sum_i \text{symp}\widehat{O}_{S\phi_i} \quad (62)$$

where  $H(\phi), P_{\phi_i}(\phi)$  can be computed as usually (see [31] for details). In this way, we have succeeded in computing all the *symp* of the observables of  $\widehat{\mathcal{L}}_S$  up to  $0(\hbar^2)$ , which are just the  $O_{\phi_i}(H(\phi), P(\phi))$ , and we have defined the mapping

<sup>22</sup>We are using the ‘final pointer basis’ of section III.C. Below we will write all the formulae in this basis.

<sup>23</sup>These polynomials have several variables, but there is no problem since all these variables commute.

$$\text{ symb} : \widehat{\mathcal{L}}_S \rightarrow \mathcal{L}_{Sq} \quad \text{ symb} \widehat{O}_S = O_S(\phi) = \sum_i O_{\phi_i}(H(\phi), P_{\phi_i}(\phi)) + 0(\hbar^2) \quad (63)$$

Moreover, since decompositions  $D_{\phi_i}$  or  $\mathcal{D}_{\phi_i}$  are arbitrary (because they depend on the initial conditions of Section III.A), from eqs.(60) and (62) we obtain (up to  $0(\hbar^2)$ )

$$\widehat{O}_{S\phi_i} = O_{\phi_i}(\widehat{H}, \widehat{P}_{\phi_i}), \quad O_{S\phi_i}(\phi) = \text{ symb} \widehat{O}_{S\phi_i} = O_{\phi_i}(H(\phi), P_{\phi_i}(\phi)) \quad (64)$$

Let us observe that, if  $O_{\phi_i}(\omega, p) = \delta(\omega - \omega')\delta(p - p')$ , we have (also up to  $0(\hbar^2)$ )

$$\text{ symb}|\omega', p'\rangle_{\phi_i} \langle \omega', p'|_{\phi_i} = \delta(H(\phi) - \omega')\delta(P_{\phi_i}(\phi) - p) \quad (65)$$

an equation that we will use below.

Summing up, from eqs.(2) and (63) we have defined a classical space  $\mathcal{A}_q = \mathcal{L}_R \oplus \mathcal{L}_S$  and a mapping

$$\text{ symb} : \widehat{\mathcal{A}} \rightarrow \mathcal{A}_q \quad \text{ symb} \widehat{O} = O(\phi) \quad (66)$$

where eqs.(9) and (10) are also valid. Then, we can repeat what we have said below eq.(10), but now for the algebra  $\widehat{\mathcal{A}}_q$  defined as in this section, with its *regular and singular parts*.

If now we take the limit  $\hbar \rightarrow 0$ , we obtain  $\mathcal{A}_q \rightarrow \mathcal{A}$ , where  $\mathcal{A}$  is the usual algebra of observables on phase space. Then, in this limit we have a correspondence  $\widehat{\mathcal{A}} \rightarrow \mathcal{A}$ . However, even if this limit is well defined and can be considered as *the classical limit of the algebra of operators*, it is only the limit of the *equations* of the system, since these are a consequence of the algebra. Therefore, this is just a 'formal' limit. The limit  $\hbar \rightarrow 0$  will be completely studied when we deal with the state space.

For the sake of simplicity, from now on we will systematically eliminate all the  $0(\hbar^2)$  from the equations and call the  $\mathcal{A}_q$  just  $\mathcal{A}$ . This is a rigorous simplification. In fact, when  $\hbar = 0$  we can make the  $0(\hbar) = 0$  everywhere since, from eq.(7), when  $\hbar = 0$  we have  $\exp 0 = 1$  in that equation; in other words, the  $\lim_{\hbar \rightarrow 0}$  is continuous.

## B. Quantum and classical states

a.- Let us remember that  $|\omega, p\rangle_{\phi_i} = |\omega, p, p\rangle_{\phi_i} = |\omega, p\rangle_{\phi_i} \langle \omega, p|_{\phi_i}$  and  $|\omega, \omega', p, p'\rangle_{\phi_i} = |\omega, p\rangle_{\phi_i} \langle \omega', p'|_{\phi_i}$  as in eq.(41).  $\{|\omega, p, p'\rangle_{\phi_i}\}$  is the basis of  $\widehat{\mathcal{L}}_S$  and  $\{|\omega, \omega', p, p'\rangle_{\phi_i}\}$  is the basis of  $\widehat{\mathcal{L}}_R$ . Then, eq.(58) reads

$$\begin{aligned} \widehat{O} &= \sum_i \widehat{O} = \sum_i \int_{p \in D_{\phi_i}} dp^N \int_0^\infty O_{\phi_i}(\omega, p) |\omega, p\rangle_{\phi_i} d\omega \\ &+ \sum_i \int_{p \in D_{\phi_i}} \int_{p' \in D_{\phi_i}} dp^N dp'^N \int_0^\infty \int_0^\infty O_{\phi_i}(\omega, \omega', p, p') |\omega, \omega', p, p'\rangle_{\phi_i} d\omega d\omega' \end{aligned} \quad (67)$$

Since the states are functionals over the space  $\widehat{\mathcal{A}} = \widehat{\mathcal{L}}_S \oplus \widehat{\mathcal{L}}_R$ , let us consider the dual space  $\widehat{\mathcal{A}}' = \widehat{\mathcal{L}}_S' \oplus \widehat{\mathcal{L}}_R'$ . We will call  $\{(\omega, p|_{\phi_i})\}$  the local bases of  $\widehat{\mathcal{L}}_S'$  and  $\{(\omega, \omega', p, p'|_{\phi_i})\}$  the local bases of  $\widehat{\mathcal{L}}_R'$ . Let us remember that  $(\omega, \omega', p, p'|_{\phi_i} = |\omega, p\rangle_{\phi_i} \langle \omega', p'|_{\phi_i}$  but  $(\omega, p|_{\phi_i} \neq |\omega, p\rangle_{\phi_i} \langle \omega, p|_{\phi_i}$ . Moreover, as in eq.(42),

$$\begin{aligned} (\omega, p|_{\phi_i} |\omega', p'\rangle_{\phi_i})_{\phi_j} &= \delta(\omega - \omega')\delta^N(p - p')\delta_{ij} \quad (\omega, \sigma, p, s|_{\phi_i} |\omega', \sigma', p', s'\rangle_{\phi_j})_{\phi_j} = \delta(\omega - \omega')\delta(\sigma - \sigma')\delta^N(p - p')\delta^N(s - s')\delta_{ij} \\ (\omega, \sigma, |_{\phi_i} |\omega', \sigma', p', s'\rangle_{\phi_j})_{\phi_j} &= (\omega, \sigma, p, s|_{\phi_i} |\omega', \sigma'\rangle_{\phi_i})_{\phi_i} = 0 \end{aligned} \quad (68)$$

Then, a generic functional of  $\widehat{\mathcal{A}}'$  reads

$$\begin{aligned} \widehat{\rho} &= \sum_i \int_{p \in D_{\phi_i}} dp^N \int_0^\infty \overline{\rho_{\phi_i}(\omega, p)} (\omega, p|_{\phi_i} d\omega \\ &+ \sum_i \int_{p \in D_{\phi_i}} \int_{p' \in D_{\phi_i}} dp^N dp'^N \int_0^\infty \int_0^\infty \overline{\rho_{\phi_i}(\omega, \omega', p, p')} (\omega, \omega', p, p'|_{\phi_i} d\omega d\omega' \end{aligned} \quad (69)$$

Like functions  $O_{\phi_i}(\omega, \omega', p, p')$ , functions  $\rho_{\phi_i}(\omega, \omega', p, p')$  are regular and have all the mathematical properties necessary to make the formalism successful (see [24]). Moreover, the  $\hat{\rho}$  must be self-adjoint, and their diagonal  $\rho_{\phi_i}(\omega, p)$  must represent probabilities; thus,  $\sum_{i,p} \int_0^\infty \rho_{\phi_i}(\omega, p) d\omega = 1$  (as in eq.(46)) and, *most important*,

$$\rho_{\phi_i}(\omega, p) \geq 0 \quad (70)$$

The  $\hat{\rho}$  with such properties belong to a convex set  $\hat{\mathcal{S}}$ , the set of states. Also, as in eq.(48),

$$\begin{aligned} (\hat{\rho}|\hat{O}) &= \sum_i \int_{p \in D_{\phi_i}} \int_0^\infty \overline{\rho_{\phi_i}(\omega, p,)} O_{\phi_i}(\omega, p) d\omega dp^N \\ &+ \sum_i \int_{p \in D_{\phi_i}} \int_{p' \in D_{\phi_i}} \int_0^\infty \int_0^\infty \overline{\rho_{\phi_i}(\omega, \omega', p, p')} O_{\phi_i}(\omega, \omega', p, p') d\omega d\omega' d^N p d^N p' \end{aligned} \quad (71)$$

b.- Since  $\hat{\mathcal{L}}_R$  and  $\hat{\mathcal{L}}'_R$  are spaces of operators on a Hilbert space  $\mathcal{H}$ , the symbol for any  $\hat{\rho}_R \in \hat{\mathcal{L}}'_R$  is defined as in eq.(15).<sup>24</sup> From this definition, eq.(16) can be proved for the regular parts (see the demonstration in [43], eq.(2.13)):

$$(\hat{\rho}_R|\hat{O}_R) = (\text{symp}\hat{\rho}_R|\text{symp}\hat{O}_R) = \sum_i \int_{D_{\phi_i}} d\phi^{2(N+1)} \rho_{\phi_i R}(\phi) O_{\phi_i R}(\phi) \quad (72)$$

Then, in  $\hat{\mathcal{L}}_R$  and  $\hat{\mathcal{L}}'_R$  all the equations are the usual ones (i.e. those of papers [43] and [44]).

Let us now consider the singular dual space  $\hat{\mathcal{L}}'_S$ , the case not treated in the bibliography. In this space we will define  $\text{symp}\hat{\rho}_S$  as the function on  $\mathcal{M}$  that satisfies an equation similar to eqs.(16) or (72) for any  $\hat{O}_S \in \hat{\mathcal{L}}_S$ , namely,

$$(\hat{\rho}_S|\hat{O}_S) \doteq (\text{symp}\hat{\rho}_S|\text{symp}\hat{O}_S)$$

precisely,

$$\sum_i \int_{p \in D_{\phi_i}} \int_0^\infty \rho_{\phi_i}(\omega, p, ) O_{\phi_i}(\omega, p) d\omega dp^N = \sum_i \int_{D_{\phi_i}} d\phi^{2(N+1)} \rho_{\phi_i S}(\phi) O_{\phi_i S}(\phi) \quad (73)$$

where the unknown density function  $\rho_S(\phi) = \text{symp}\hat{\rho}_S$  can be decomposed as

$$\text{symp}\hat{\rho}_S = \rho_S(\phi) = \sum_i \rho_{\phi_i S}(\phi) \quad (74)$$

in each  $D_{\phi_i}$ . Thus, since we know  $\rho_{\phi_i}(\omega, p, )$ ,  $O_{\phi_i}(\omega, p)$ , and  $O_{\phi_i S}(\phi)$ , we can compute  $\rho_{\phi_i S}(\phi)$  to obtain  $\rho_S(\phi) = \text{symp}\hat{\rho}_S$ . Now (as suggested by eq.(62)),  $\hat{\rho}_S$ , being time invariant, must be a function of the constants of the motion; therefore (as in Subsection A) its Weyl-transformed  $\rho_S(\phi)$  must be endowed with the same property, but now in the classical case. Since the  $\{H(\phi), P_{\phi_i}(\phi)\}$  are locally a complete set of constants of the motion in involution, we must have

$$\rho_{\phi_i}(\phi) = F(H(\phi), P_{\phi_i}(\phi)) \quad (75)$$

We will find the function  $F$ . The system has a local pointer CSCO of  $N + 1$  operators and the dimension of its phase space is  $2(N + 1)$ , i.e. it is *locally* an integrable system.<sup>25</sup> Then, we can define *locally at*  $D_{\phi_i}$  the action angle variables  $(\theta^0, \theta^1, \dots, \theta^N, J_{\phi_i}^0, J_{\phi_i}^1, \dots, J_{\phi_i}^N)$ , where  $J_{\phi_i}^0, J_{\phi_i}^1, \dots, J_{\phi_i}^N$  would be just  $H, P_{\phi_{i1}}, \dots, P_{\phi_{iN}}$  (multiplied by adequate constants in such a way that the  $\theta_{\phi_i}^{i0}$  variables belong to an interval  $0 \leq \theta_{\phi_i}^{i0} \leq 2\pi$  in the integrable case). We will call

<sup>24</sup>We repeat that, in the case of states, we must add a new factor  $(2\pi\hbar)^{-(N+1)}$  to definition (4) in order to maintain the usual normalization of  $\rho(\phi)$ .

<sup>25</sup>We have discussed this fact in detail at the beginning of section III. The constants  $J$  are global or isolating in the case of an integrable system, but not in the non-integrable case. Nevertheless, they are locally defined. Moreover, we will only consider the cases where action-angle variables can be locally defined.



'J' just the ' $H, P_{\phi_i}$ '. Thus, we can make the canonical transformation  $\phi^a \rightarrow \theta_{\phi_i}^0, \theta_{\phi_i}^1, \dots, \theta_{\phi_i}^N, H, P_{\phi_i 1}, \dots, P_{\phi_i N}$ , and we obtain

$$d\phi^{2(N+1)} = dq^{(N+1)} dp^{(N+1)} = d\theta_{\phi_i}^{(N+1)} dH dP_{\phi_i}^N \quad (76)$$

because the Jacobian of a canonical transformation is  $\pm 1$ . Since the transformation must keep the metric of eq.(3), we can take 1 with no loss of generality.

In order to compute the l.h.s. of eq.(73), we must know how to integrate functions  $f(H, P_{\phi_i}) = f(H, P_{\phi_i}, \dots, P_{\phi_i})$  which are just functions of the constant of motion, precisely,

$$\begin{aligned} \int_{D_{\phi_i}} d\phi^{2(N+1)} f(H, P_{\phi_i}) &= \int_{D_{\phi_i}} d\theta_{\phi_i}^{(N+1)} dH dP_{\phi_i}^N f(H, P_{\phi_i}) \\ &= \int_{D_{\phi_i}} dH dP_{\phi_i}^N C_{\phi_i}(H, P_{\phi_i}) f(H, P_{\phi_i}) \end{aligned} \quad (77)$$

where we have integrated the angular variables  $\theta_{\phi_i}^0, \theta_{\phi_i}^1, \dots, \theta_{\phi_i}^N$  and obtained the configuration volume  $C_{\phi_i}(H, P_{\phi_i})$  of the portion of the hypersurface defined by ( $H = \text{const.}, P_{\phi_i} = \text{const.}$ ) and contained in  $D_{\phi_i}$ . So, from eqs.(73) and (77) we have that

$$\begin{aligned} \int_{p \in D_{\phi_i}} \int_0^\infty \rho_{\phi_i}(\omega, p, ) O_{\phi_i}(\omega, p) d\omega dp^N \\ = \int dH dP_{\phi_i}^N C_{\phi_i}(H, P_{\phi_i}) \rho_{\phi_i S}(H, P_{\phi_i}) O_{\phi_i S}(H, P_{\phi_i}) \end{aligned} \quad (78)$$

for all  $O_{\phi_i}(H, P_{\phi_i}) = O_{S\phi_i}(H, P_{\phi_i})$  (see eq.(64)). The last equation defines  $\rho_{S\phi_i}(H, P) = \frac{1}{C_{\phi_i}} \rho_{\phi_i}(H, P)$  for  $\phi \in \mathcal{D}_{\phi_i}$ ,<sup>26</sup> but not for  $\phi \in \mathcal{M} \setminus \mathcal{D}_{\phi_i}$ <sup>27</sup>; then, as in the case of  $O_{S\phi_i}(\phi)$ , we will consider that  $\rho_{S\phi_i}(\phi) = 0$  for  $\phi \in \mathcal{M} \setminus \mathcal{D}_{\phi_i}$  and that they are defined all over  $\mathcal{M}$  (this causes no problem because  $O_{S\phi_i}(\phi)$  is multiplied by  $\rho_{S\phi_i}(\phi)$ , and  $O_{S\phi_i}(\phi)$  has this property). In this way, we can arrive from eq.(74) to our final result

$$\rho_S(\phi) = \rho_*(\phi) = \sum_i \frac{1}{C_{\phi_i}(H, P_{\phi_i})} \rho_{\phi_i}(H(\phi), P_{\phi_i}(\phi)) \quad (79)$$

Now, from eq.(70) we obtain that

<sup>26</sup>In the integrable case, where there is just one  $\rho(H, P)$ , it would be  $\rho_{\phi_i}(H, P) = \frac{C_{\phi_i}(H, P)}{2\pi^{N+1}} \rho(H, P)$  and the results of paper [26] would be reobtained. In fact, integrating over a torus in the  $\theta$  we have  $(2\pi)^{N+1} \rho(H, P) = \sum_i C_{\phi_i}(H, P) \rho_{\phi_i}(H, P)$ .

An example to fix the ideas: let us consider the harmonic oscillator and the plane  $q, p$  in radial coordinates  $\theta, H$ . Let us define two  $D_{\phi_i}$ :  $D_1$  with  $0 \leq \theta < \Theta(H)$  and  $D_2$  with  $\Theta(H) \leq \theta < 2\pi$ , where  $\Theta(H)$  is an arbitrary function. Then,

$$\rho(\phi) = \rho_1(\phi) I_1(\phi) + \rho_2(\phi) I_2(\phi)$$

If  $\rho(\phi) = \rho(H)$ , by integrating over the  $\theta$  we obtain

$$\begin{aligned} 2\pi \rho(H) &= \int_0^{\Theta(H)} \rho_1(H) I_1(\phi) d\theta + \int_{\Theta(H)}^{2\pi} \rho_2(H) I_2(\phi) d\theta \\ &= \rho_1(H) \Theta(H) + (2\pi - \Theta(H)) \rho_2(H) = \\ &\quad \rho_1(H) C_1(\phi) + \rho_2(H) C_2(\phi) \end{aligned}$$

namely, the equation  $(2\pi)^{N+1} \rho(H, P) = \sum_i C_{\phi_i}(H, P) \rho_{\phi_i}(H, P)$  for this particular case with dimension  $N + 1 = 1$

<sup>27</sup>We will forget the joining zones  $\mathcal{F}_{\phi_i}$  and  $F_{\phi_i}$  since we have already proved that, when  $S \rightarrow \infty$ , their influence is irrelevant.

$$\rho_S(\phi) = \rho_*(\phi) \geq 0 \quad (80)$$

This means that the Wigner transformation of the singular part *can be considered a density function since it is non-negatively defined* (of course, this is not the case for the regular part).

Always working in the domain  $\mathcal{D}_{\phi_i}$  and making  $\rho_{\phi_i}(\omega, p) = \delta(\omega - \omega')\delta^N(p - p')$ , we also have<sup>28</sup>

$$\text{symp}(\omega', p', (\phi)|_{\phi_i}) = \frac{1}{C_{\phi_i}(H, P_{\phi_i})} \delta(H(\phi) - \omega') \delta^{(N)}(P(\phi) - p'_{\phi_i}) \quad (81)$$

c.- From eqs.(56) and (81) we obtain

$$\rho_S(\phi) = \rho_*(\phi) = \sum_i \int_{p \in D_{\phi_i}} dp \int_0^\infty \rho_{\phi_i}(\omega, p) \frac{1}{C_{\phi_i}(H, P_{\phi_i})} \delta(H(\phi) - \omega) \delta^{(N)}(P(\phi) - p_{\phi_i}) d\omega \quad (82)$$

The continuity of the function  $\rho_*(\phi)$ , when it goes from one  $D_{\phi_i}$  to another  $D_{\phi_j}$  ( $i \neq j$ ), is demonstrated in Section V. Therefore, we have obtained a decomposition of  $\rho_*(\phi) = \rho_S(\phi)$  in classical hypersurfaces ( $H = \omega$ ,  $P_{\phi_i}(\phi) = p_{\phi_i}$ ), containing classical trajectories, summed with different positive weight coefficients  $\rho_{\phi_i}(\omega, p)/C_{\phi_i}(H, P_{\phi_i})$ , and represented in different ways in each domain  $D_{\phi_i}$ , but still with the same interpretation as in the integrable case.<sup>29</sup>

d.- Since now we know how to deal with the singular part, we have defined the mapping of the quantum space of states  $\mathcal{A}'$  on the 'classical' space of states  $\mathcal{A}'$

$$\text{symp} : \widehat{\mathcal{A}'} \rightarrow \mathcal{A}' \quad (83)$$

In the limit  $\hbar \rightarrow 0$ , eqs.(9) and (10) are always valid; then, it might be supposed that we have arrived to the classical limit for the states. But *this is not so* because, in general, even for  $S$  very big (or  $\hbar$  very small) the obtained  $\rho(\phi)$  *does not satisfy* the condition (see also the Appendix of ref. [31])

$$\rho(\phi) \geq 0 \quad (84)$$

This is due to the fact that the regular part is still present and this part does not satisfy the last condition (on the contrary, from eq.(80) we can see that the singular part satisfies the last inequality). As a consequence,  $\rho(\phi)$  is not a density function and, therefore, the mapping (83) is not a mapping of quantum mechanics on classical statistical mechanics. This mapping does not give us the classical world, but a deformed classical world where 'density functions' can be negative. In other words, when  $\hbar \rightarrow 0$  the isomorphism (83) is a mapping of quantum mechanics on a certain quantum mechanics 'alla classica', namely, only formulated in phase space  $\mathcal{M}$  but not satisfying eq. (84). This clearly shows that  $\hbar \rightarrow 0$  is not the classical limit. In order to obtain this limit, we have to introduce decoherence, as previously studied, both at the quantum and the classical level.

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<sup>28</sup>In the chaotic, homogeneous, ergodic case, we have a  $N + 1$ -CSCO with just  $\hat{H}$  and, classically, just  $H$  as a constant of motion. In this case (see [55], p.247),

$$\rho_S(\phi) = \rho_*(\phi) = \int_0^\infty \delta(\omega - E) \frac{1}{C(H)} \delta(H(\phi) - E) = \frac{\delta(H(\phi) - E)}{\int dq dp \delta(H(\phi) - E)}$$

<sup>29</sup>We can verify the normalization:

$$\begin{aligned} \int \rho_S(\phi) d\phi^{2(N+1)} &= \sum_i \int \rho_S(\phi) dH dP_{\phi_i}^N d\theta_{\phi_i}^{N+1} = \\ &= \sum_i \int dH dP_{\phi_i}^N \frac{\rho_S(H, P_{\phi_i})}{C_{\phi_i}(H, P_{\phi_i})} \int d\theta_{\phi_i}^{N+1} = \\ &= \sum_i \int dH dP_{\phi_i}^N \rho_S(H, P_{\phi_i}) = 1 \end{aligned}$$

### C. Time evolution and decoherence

As we have seen, the only thing that prevents us from having a good isomorphism (83) is that the regular parts do not satisfy condition (84). But we know from eqs.(50) or (56) that, for  $t \rightarrow \infty$ , the regular part vanishes and only the singular part remains, which does satisfy this condition. As a consequence, after decoherence and  $\hbar \rightarrow 0$  (that is, the elimination of all the  $0(\hbar^2)$  that we have omitted), we finally obtain the classical statistical limit since the classical densities obtained obey all the laws of classical statistical mechanics. In fact, as we will see in the next section in detail, eq.(82) shows that these distributions are the result of classical point-like-states moving in phase space and following classical trajectories. The usual classical limit is obtained by choosing one of these trajectories; we will explain this procedure in the next section.

### V. THE CLASSICAL LIMIT

From what we have learnt above, we can explain with more detail the three steps involved in the classical limit, presented in the introduction and shown in the following diagram:

*Quantum Mechanics – (decoherence)  $\rightarrow$  Boolean Quantum Mechanics – (symp and  $\hbar \rightarrow 0$ )  $\rightarrow$*

*Classical Statistical Mechanics – (choice of a trajectory)  $\rightarrow$  Classical Mechanics*

Let us comment these three steps:

i.- *Quantum Mechanics – (decoherence)  $\rightarrow$  Boolean Quantum Mechanics.* Decoherence transforms non-Boolean quantum mechanics into Boolean quantum mechanics<sup>30</sup> since it eliminates the off-diagonal terms, as we have shown in eq.(56).

ii.- *Boolean Quantum Mechanics – (symp and  $\hbar \rightarrow 0$ )  $\rightarrow$  Classical Statistical Mechanics.* The Wigner transformation *symp* and the limit  $\hbar \rightarrow 0$  are defined with no problems in the singular part remaining after decoherence. They map Boolean quantum mechanics onto classical statistical mechanics: this is what we have essentially shown above. Our demonstration culminates in Section IV.C, where we have proved that the transformed quantum Boolean states are really positively defined densities. From eq.(82) we also know that these densities are the sums of densities strongly peaked on the classical hypersurfaces defined by the constants of the motion  $H(\phi) = \omega$ ,  $P_{\phi_i}(\phi) = p_{\phi_i}$ . In the next step we will see that such classical hypersurfaces contain classical trajectories averaged by the coefficients  $\rho_{\phi_i}(\omega, p)$ .

iii.- *Classical Statistical Mechanics – (choice of a trajectory)  $\rightarrow$  Classical Mechanics (Localization or Actualization).* After step (ii), we are still in classical statistical mechanics but not in proper classical mechanics. To perform the last step we have to pass from classical densities to classical trajectories (i.e. to consider the localization effect<sup>31</sup>). For this purpose let us observe that, after the two first steps, the formalism of Boolean quantum mechanics is isomorphic with the formalism of statistical classical mechanics:

- *For the observables:* After *symp* and  $\hbar \rightarrow 0$ , we obtain the correspondence  $\text{symp} : \widehat{\mathcal{A}} \sim \mathcal{A}$  (see Section IV A), namely,

$$A_{\phi_i}(\widehat{H}, \widehat{P}_{\phi_i}) \sim A_{\phi_i}(H(\phi), P_{\phi_i}(\phi))$$

- *For the states:* After decoherence, *symp* and  $\hbar \rightarrow 0$ , again  $\text{symp} : \widehat{\mathcal{A}}' \sim \mathcal{A}'$  (see Section IV.B), namely,

$$\rho_{\phi_i}(\widehat{H}, \widehat{P}_{\phi_i}) \sim \rho_{\phi_i}(H(\phi), P_{\phi_i}(\phi)) \geq 0$$

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<sup>30</sup>Namely, quantum mechanics in the local CSCO  $\{\widehat{H}, \widehat{P}\}$  using only diagonal states.

<sup>31</sup>See [47], Chap. 4, for a different view.

and the states  $\rho_*(\widehat{H}, \widehat{P})$  and  $\rho_*(H(\phi), P(\phi))$  are time invariant:

$$\begin{aligned}\rho_*(\widehat{H}, \widehat{P}) &= \sum_i \int d\omega \int_{p \in D_{\phi_i}} dp^N \frac{\rho_{\phi_i}(\omega, p)}{C(\omega, p)} (\omega, p|_{\phi_i} \\ \rho_*(H(\phi), P(\phi)) &= \sum_i \int d\omega \int_{p \in D_{\phi_i}} dp^N \frac{\rho_{\phi_i}(\omega, p)}{C_{\phi_i}(\omega, p)} \delta(H(\phi) - \omega) \delta^N(P_{\phi_i}(\phi) - p_{\phi_i})\end{aligned}\quad (85)$$

Moreover, since  $\Delta A \Delta B \geq \frac{\hbar}{2} |\langle [A, B] \rangle_\rho|$ , in the limit  $\hbar \rightarrow 0$  there are no uncertainty relations and the algebras  $\widehat{\mathcal{A}}$  and  $\mathcal{A}$  can be considered commutative (remember that, according to the uncertainty principle,  $\hbar \rightarrow 0$  has the same effect that  $[A, B] = 0$ ). In other words, in the limit  $\hbar \rightarrow 0$  all the picture is classical in such a way that the trajectories, contained in the hypersurfaces  $H(\phi) = \omega$ ,  $P_{\phi_i}(\phi) = p_{\phi_i}$ , could be interpreted as *real classical trajectories*. However, the  $\delta(H(\phi) - \omega) \delta^N(P_{\phi_i}(\phi) - p_{\phi_i})$  still represent states strongly peaked around these hypersurfaces (but not around trajectories). Therefore, if we want to obtain an equation like (85) but clearly showing the classical trajectories, we have to introduce the initial conditions of each trajectory.

Let us consider a classical trajectory in phase space  $\mathcal{M} = \mathcal{M}_{2(N+1)}$ , expressed in the classical coordinates  $(\tau, \theta_{\phi_i}, H, P_{\phi_i})$ , where  $\tau$  is the coordinate canonically conjugated to  $H$  and the  $\theta_{\phi_i}$  are the coordinates canonically conjugated to the  $P_{\phi_i}$ . The constants of the motion in involution are  $\{H, P_{\phi_i}\}$ ; but, for conciseness and generality, let us consider that the constants of the motion in involution are  $\{\Pi_{\phi_i}\}$  with conjugated coordinates  $\{A_{\phi_i}\}$ , and that  $H = H(\Pi_{\phi_i})$ . From the von Neumann-Liouville equation in the Heisenberg representation,

$$i\hbar \frac{d\widehat{A}}{dt} = [A, H]$$

we obtain

$$\frac{dA(\phi)}{dt} = \{A, H\}_{mb} = \{A, H\}_{pb} + 0(\hbar^2)$$

Then, the Hamiltonian equations in the limit  $\hbar \rightarrow 0$  read<sup>32</sup>

$$\frac{dA_{\phi_i}}{dt} = \frac{\partial H}{\partial \Pi_{\phi_i}} = \Omega_{\phi_i}(\Pi_{\phi_i}) = \text{const.}; \quad \frac{d\Pi_{\phi_i}}{dt} = -\frac{\partial H}{\partial A_{\phi_i}} = 0 \quad (86)$$

The classical trajectories are

$$A_{\phi_i}(t) = A_{\phi_i}^{(0)} + \Omega_{\phi_i}(\Pi_{\phi_i})t, \quad \Pi_{\phi_i} = \Pi_{\phi_i}^{(0)} = \text{const.} \quad (87)$$

where the  $A_{\phi_i}^{(0)}$  and  $\Pi_{\phi_i}^{(0)}$  are integration constants. A distribution strongly peaked on this trajectory reads

$$\delta[A_{\phi_i}(t) - A_{\phi_i}^{(0)} - \Omega_{\phi_i}(\Pi_{\phi_i})t] \delta(\Pi_{\phi_i} - \Pi_{\phi_i}^{(0)})$$

and a general classical distribution evolving according to the motion (87) reads<sup>33</sup>

$$\begin{aligned}\rho_C(t, \phi) &= \sum_i \int_{D_{\phi_i}} \rho_{\phi_i}^{(C)}(A_{\phi_i}^{(0)}, \Pi_{\phi_i}^{(0)}) \delta[A_{\phi_i}(t) - A_{\phi_i}^{(0)} - \Omega_{\phi_i}(\Pi_{\phi_i})t] \times \\ &\quad \delta(\Pi_{\phi_i} - \Pi_{\phi_i}^{(0)}) d^{N+1} A_{\phi_i}^{(0)} d^{N+1} \Pi_{\phi_i}^{(0)}\end{aligned}\quad (88)$$

where  $\rho_{\phi_i}^{(C)}(A_{\phi_i}^{(0)}, \Pi_{\phi_i}^{(0)})$  is a generic classical coefficient (undefined up to now). If we want that this density (evolving according to a Frobenius-Perron evolution, see [56]) be an equilibrium density, we have to eliminate the variable  $t$ .

<sup>32</sup>These equations correspond to the system of differential equations (3.1) of [56].

<sup>33</sup>If the evolution  $S^t$  of [56] were the (87), the corresponding density would be  $f(t, x) \equiv P^t f(x)$  (see [56], eq.(3.2)) where  $P^t$  would represent a Frobenius-Perron evolution. Moreover, it is easy to show that  $\rho(t, \phi)$  satisfies the Liouville equation.

For this purpose, it is sufficient to choose the initial distribution  $\rho_{\phi_i}^{(C)}(A_{\phi_i}^{(0)}, \Pi_{\phi_i}^{(0)})$  as just a function of  $\Pi_{\phi_i}^{(0)}$ , namely,  $\rho_{\phi_i}^{(C)}(\Pi_{\phi_i}^{(0)})$ , which is still a free function that we can use to represent different  $\rho_C(\phi)$ . Then, we obtain

$$\rho_C(\phi) = \sum_i \int_{D_{\phi_i}} \rho_{\phi_i}^{(C)}(\Pi_{\phi_i}^{(0)}) \delta(\Pi_{\phi_i} - \Pi_{\phi_i}^{(0)}) d^{N+1} \Pi_{\phi_i}^{(0)} \quad (89)$$

since, for any fixed  $t$ , we have

$$\sum_i \int_{D_{\phi_i}} \delta(A_{\phi_i}(t) - A_{\phi_i}^{(0)} - \Omega_{\phi_i}(\Pi_{\phi_i})t) d^{N+1} A_{\phi_i}^{(0)} = 1$$

Going back to our primitive variables, eq.(88) reads

$$\begin{aligned} \rho_C(\phi) &= \sum_i \int \rho_{\phi_i}^{(C)}(\omega, p) \delta(H(\phi) - \omega) \delta(P_{\phi_i} - p_{\phi_i}) \\ &\times \delta(\tau(\phi) - \tau_0 - \omega t) \delta(\theta_{\phi_i}(\phi) - \theta_{\phi_i 0} - p_{\phi_i} t) d\omega d^N p_{\phi_i} d\tau_0 d\theta_{\phi_i 0} \end{aligned} \quad (90)$$

while eq.(89) reads

$$\rho_C(\phi) = \sum_i \int \rho_{\phi_i}^{(C)}(\omega, p) \delta(H(\phi) - \omega) \delta(P_{\phi_i} - p_{\phi_i}) d\omega d^N p_{\phi_i}$$

Then, from eq.(85) and making the undefined coefficient  $\rho_{\phi_i}^{(C)}(\omega, p) = \frac{\rho_{\phi_i}(\omega, p)}{C_{\phi_i}(\omega, p_{\phi_i})}$ , we have

$$\rho_*(\phi) = \rho_C(\phi) \quad (91)$$

The function  $\rho_C(\phi)$  can be interpreted as the equilibrium density of a Frobenius-Perron evolution of particle-like states  $(\tau, \theta_{\phi_i}, H, P_{\phi_i})$ , as if these states would move in phase space  $\mathcal{M} = \mathcal{M}_{2(N+1)}$  according to the classical motions (87).

However, each term of the sum  $\sum_i$  of eq.(90) is valid in the chart  $\mathcal{D}_{\phi_i}$  ( $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$ ). In a different chart  $\mathcal{D}_{\phi_j}$  ( $D_{\phi_j} \subset \mathcal{D}_{\phi_j}$ ), the equation is also valid and, then, at  $\phi \in \mathcal{D}_{\phi_i} \cap \mathcal{D}_{\phi_j}$  we have

$$\begin{aligned} &\int \frac{\rho_{\phi_i}(\omega, p)}{C_{\phi_i}(\omega, p_{\phi_i})} \delta(H(\phi) - \omega) \delta(P_{\phi_i} - p_{\phi_i}) \\ &\times \delta(\tau(\phi) - \tau_0 - \omega t) \delta(\theta_{\phi_i}(\phi) - \theta_{\phi_i 0} - p_{\phi_i} t) d\omega d^N p_{\phi_i} d\tau_0 dA_{\phi_i 0} = \\ &\int \frac{\rho_{\phi_i}(\omega, p)}{C_{\phi_i}(\omega, p_{\phi_j})} \delta(H(\phi) - \omega) \delta(P_{\phi_j} - p_{\phi_j}) \\ &\times \delta(\tau(\phi) - \tau_0 - \omega t) \delta(\theta_{\phi_j}(\phi) - \theta_{\phi_j 0} - p_{\phi_j} t) d\omega d^N p_{\phi_j} d\tau_0 dA_{\phi_j 0} \end{aligned}$$

Here it is worth emphasizing that the trajectories  $H = \omega$ ,  $P_{\phi_i}(\phi) = p_{\phi_i}$ ,  $\tau(\phi) = \tau_0 + \omega t$ ,  $\theta_{\phi_i}(\phi) = \theta_{\phi_i 0} + p_{\phi_i} t$  in chart  $\mathcal{D}_{\phi_i}$  are *continuously connected* with those  $H = \omega$ ,  $P_{\phi_j}(\phi) = p_{\phi_j}$ ,  $\tau(\phi) = \tau_0 + \omega t$ ,  $\theta_{\phi_j}(\phi) = \theta_{\phi_j 0} + p_{\phi_j} t$  in chart  $\mathcal{D}_{\phi_j}$ , because these charts are not generic but constructed using the solution of eqs.(19), (23) or (86). Since  $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$  and  $D_{\phi_j} \subset \mathcal{D}_{\phi_j}$ , the same holds for the trajectories going from  $D_{\phi_i}$  to  $D_{\phi_j}$ . Thus, the continuous connection follows from the fact that *one and only one* solution of the trajectory equation passes for each point of  $\mathcal{M}$  (and, therefore, for each  $\phi \in \mathcal{D}_{\phi_i} \cap \mathcal{D}_{\phi_j}$ ).

Summing up, we have obtained a decomposition of  $\rho_*(\phi) = \rho_S(\phi)$  in classical trajectories  $H = \omega$ ,  $P_{\phi_i}(\phi) = p_{\phi_i}$ ,  $\tau(\phi) = \tau_0 + \omega t$ ,  $\theta_{\phi_j}(\phi) = \theta_{\phi_j 0} + p_{\phi_j} t$ , summed with different weight coefficients  $\rho_{\phi_i}(\omega, p)/C_{\phi_i}(H, P_{\phi_i})$  and represented in different ways in each domain  $\mathcal{D}_{\phi_i}$ , but still with the same interpretation as in the integrable case. Moreover, as announced in Section III.A.c, we see that chart  $\mathcal{D}_{\phi_i}$  is continuously connected with chart  $\mathcal{D}_{\phi_j}$ , for any  $\mathcal{D}_{\phi_i}$ ,  $\mathcal{D}_{\phi_j}$ . Therefore, we have finally obtained the classical limit to the extent that we have described each one of the classical trajectories. But, since from the very beginning our system was a non-integrable one, we have obtained *the classical limit of a non-integrable system*, where the tori are broken and the trajectories are *potentially chaotic trajectories*.

Finally, we must remark that:

- Each one of the described processes, decoherence, route to macroscopicity, i.e.  $\hbar \rightarrow 0$  (e.g. the macroscopicity obtained when the two rays of an Stern-Gerlach experiment gradually separate), and eventually localization (e.g. by a localizing potential, see [26], Appendix A), has its own characteristic time; in particular, the decoherence time is computed in [54].
- We have explained the classical limit as if each process (decoherence, macroscopicity, and localization) took place one after the other, only for didactical reasons. But this is an oversimplified picture of the phenomenon. In fact, this may be not the case if the different processes overlap. Considering that they have different characteristic times, there are different possibilities according to the order in which the processes finish.

## VI. PARTIALLY NON-INTEGRABLE SYSTEMS: THE TRACE OF THE NON-INTEGRABLE PART

In this section we will consider some well-known definitions, find some relations with the concepts of statistical physics, and discuss the trace of the non-integrable part of the system. All the definitions are formulated in the classical language, but they become quantum definitions if we translate them with the Weyl-Wigner-Moyal isomorphism. In some sense, that we will precise elsewhere, these quantum definitions correspond to a quantum chaotic hierarchy.

The classical constants of the motion  $H(\phi)$ ,  $P_1(\phi)$ , ...,  $P_N(\phi)$  can be rigorously classified as [57], [58], [59]:

i.- **Global or isolating constants of the motion**, that we will call ' $H$ ',

$$H_0(\phi) = H(\phi), \quad H_1(\phi) = P_1(\phi), \dots, H_A(\phi) = P_A(\phi) \quad (92)$$

The constants of the motion are global when the conditions

$$H_\alpha = p_\alpha, \quad \alpha = 0, \dots, A \quad (93)$$

define, for each set of constants  $(p_0, \dots, p_A)$ , a global sub-manifold  $\mathcal{M}(p_0, \dots, p_A)$  (a torus in the bounded case) of phase space where the trajectories are confined.<sup>34</sup> The dimension of  $\mathcal{M}(p_0, \dots, p_A)$  is  $2(N+1) - (A+1) = 2N - A + 1$ .

ii.- **Local or non-isolating constants of the motion**, that we will call ' $J_{\phi_i}$ ',

$$J_{\phi_i 1}(\phi) = P_{\phi_i A+1}(\phi), \dots, J_{\phi_i N-A}(\phi) = P_{\phi_i N}(\phi) \quad (94)$$

The constants of the motion are local when the conditions

$$J_{\phi_i \beta} = p_{\phi_i \beta + A} \quad \beta = 1, \dots, N - A \quad (95)$$

do *not* define any global sub-manifold, but define just a local sub-manifold at  $D_{\phi_i}$ .<sup>35</sup> The  $J_{\phi_i \beta}$  are *local momentum coordinates*.

When  $A = N$ , we say that the system is *integrable*; when  $A = 0$  we say that the system is *non-integrable* (even if we will always consider  $H$  globally defined), if  $A < N$  we will say that the system is *partially non-integrable*. Let us consider these three cases in detail.

### A. Integrable systems

In this case, all the  $P$  are isolating constants of the motion  $H$ , and there are no  $J_{\phi_i}$ . Then, condition (93) foliates the phase space with submanifolds  $\mathcal{M}(p_0, \dots, p_N)$  of dimension  $N+1$ , labelled by the constants  $(p_0, \dots, p_N)$ . These submanifolds are tori when the system is endowed with action-angle variables (e.g. when phase space is bounded). On these tori, the motion of the configuration variables is the motion described by classical mechanics. In the generic

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<sup>34</sup> Isolating constants of the motion correspond to the 'simple' constants of the motion in [60], p.60.

<sup>35</sup> Let us list the introduced dimensions:

- i.- The total dimension is  $2(N+1)$ .
- ii.- The number of the isolating constants is  $A+1$ .
- iii.- The number of the non-isolating constants is  $N-A$ .
- iv.- The number of the configuration coordinates is  $N+1$ .
- v.- The dimension of  $\mathcal{M}(p_0, \dots, p_A)$  is  $2N - A + 1$

case, the frequencies of the motions are *not rationally dependent* (or non-commensurable); thus, the corresponding trajectories fill each torus in a dense way: the motion is *ergodic* on each torus.<sup>36</sup> Moreover, we can see from eq.(79) or in paper [26] that, if the angle-action variables exist, there is a unique equilibrium state on each torus,

$$\rho_*(\phi) = \frac{1}{(2\pi)^{N+1}} \rho(p_0, p_1, \dots, p_N) \quad (96)$$

which is constant in the submanifold  $\mathcal{M}(p_0, \dots, p_N)$ ; this corresponds to a *microcanonical* equilibrium on each torus.

## B. Partially non-integrable systems

In this case, not all the  $P$  are isolating constants of the motion; so, there are  $H$  and  $J_{\phi_i}$ . Then, the trajectories must be dense in a domain  $\mathcal{D}(p_0, \dots, p_A) \subset \mathcal{M}(p_0, \dots, p_A)$  of dimension  $2N - A + 1$ . In fact, if the dimension of  $\mathcal{D}(p_0, \dots, p_A)$  were  $< 2N - A + 1$ , at least a new global constant would exist and there would be  $A + 2$  global constants; but this is impossible since, by its own definition,  $A + 1$  is the total number of these constants. It is quite clear that something as a 'thermodynamic equilibrium' classical density must be globally defined in  $\mathcal{D}(p_0, \dots, p_A)$  and, therefore, the  $J_{1\phi_0}(\phi), \dots, J_{N-A\phi_0}(\phi)$  cannot be explicit variables of  $\rho_*(\phi)$ ;<sup>37</sup> then, we must have (as in eq.(79)<sup>38</sup>)

$$\rho_*^T(\phi) = \frac{1}{C(H_0(\phi), H_1(\phi), \dots, H_A(\phi))} \rho(H_0(\phi), H_1(\phi), \dots, H_A(\phi)) \quad (97)$$

Since  $\mathcal{D}(p_0, \dots, p_A)$  is contained in the hypersurface defined by

$$H_0(\phi) = p_0 = \text{const.}, \quad H_1(\phi) = p_1 = \text{const.}, \quad \dots, \quad H_A(\phi) = p_A = \text{const.} \quad (98)$$

$C(H_0(\phi), H_1(\phi), \dots, H_A(\phi)) = C(p_0, p_1, \dots, p_A)$  is the volume of the portion of the just mentioned hypersurface contained in  $\mathcal{D}(p_0, \dots, p_A)$  (in such a way that  $\rho_*^T(\phi)$  would be normalized). Therefore,

$$\rho_*^T(\phi) = \frac{1}{C(p_0, p_1, \dots, p_A)} \rho(p_0, p_1, \dots, p_A) = \text{const.} \quad (99)$$

and we have found a unique equilibrium for each  $\mathcal{D}(p_0, \dots, p_A)$ , namely, for each set of isolating constants of the motion or state variables  $(p_0, \dots, p_A)$ . Now we can repeat the argument of the integrable case. In this case, phase space is foliated in submanifolds  $\mathcal{M}(p_0, \dots, p_A)$  of dimension  $2N - A + 1$ . The difference is that now not all the coordinates of these submanifolds are configuration variables:<sup>39</sup> since there are also momentum variables, in this case we cannot use the argument about frequencies not rationally related. Nevertheless, in each  $\mathcal{D}(p_0, \dots, p_A) \subset \mathcal{M}(p_0, \dots, p_A)$  there is a unique equilibrium state (99); as a consequence, by using theorem (4.3) of ref. [56] we can conclude that the motion is *ergodic*.<sup>40</sup>

At this point, we may ask ourselves why the density  $\rho_*(H_0(\phi), H_1(\phi), \dots, H_A(\phi), J_1(\phi), \dots, J_{N-A}(\phi))$  loses its  $J_{\phi_i}$  variables. The physical reason for this relies on the fact that the space of observables  $\hat{\mathcal{O}}$  contains only physical

<sup>36</sup>If the system is unbounded, ergodicity requires that the trajectories be dense in a domain  $\mathcal{D}(p_0, \dots, p_N) \subset \mathcal{M}(p_0, \dots, p_N)$  (see subsection B).

<sup>37</sup>Even if another kind of equilibrium could be defined using local coordinates, it is quite clear that only globally defined variables can play the role of thermodynamic variables in eq.(99).

<sup>38</sup>There are not indices  $\phi_i$  because the variables are the constants of the motion globally defined in a manifold of  $2(A + 1)$  dimensions.

<sup>39</sup>Regarding the configuration variables, it is clear that, in the non-integrable case, none of them is a global constant of the motion. This fact further reduces the dimension of  $\mathcal{M}$  (or  $\mathcal{D}$ ). In fact:

- i.- The preserved tori satisfy an irrationality condition ([58], eq. (3.4.12)); so, the corresponding ratios of the frequencies are irrational and the trajectories are dense in those tori.
- ii.- In the broken tori the trajectories are chaotic. Nevertheless, if in a particular case there is a configuration variable  $X$  that turns out to be a global constant of the motion, it can be considered among the ' $H$ '. Then, we will essentially work in the manifold  $X = \text{const.}$  and nothing will change.

<sup>40</sup>Theorem (4.3) requires that the evolution be a Frobenius-Perron one. As explained in section V, this is precisely the case. In the equilibrium  $\hbar \rightarrow 0$  limit, the motion takes place along classical trajectories. So, even if  $\rho_*(\phi)$  is constant in time, the particles of the system are in motion.

measurable observables. In the integrable case, we have a global  $N + 1$ -CSCO whose observables can be globally measured in an independent way because they commute. More generally, we can measure only the variables that belong to a global set of commuting observables, even if the number of observables is  $< N + 1$ . In fact, the  $J$  could be measured in a local  $N + 1$ -CSCO, but they change when they go from one local  $N + 1$ -CSCO to another; therefore, the period for making the measurement could be very short (eventually shorter than the period necessary for the measurement itself, turning the measurement impossible). In other words, since the classical momenta  $J_{1\phi_i}(\phi), \dots, J_{N-A\phi_i}(\phi)$  have an ergodic motion, it is reasonable to suppose that the quantum analogues  $\widehat{J_{1\phi_i}}, \dots, \widehat{J_{N-A\phi_i}}$  cannot be really measured, even at the quantum level.<sup>41</sup> This means that the set  $\{\widehat{H}_0, \dots, \widehat{H}_A\}$  is the relevant measurable global  $A + 1$ -CSCO: it is possible to measure only the  $\widehat{H}, \widehat{H}_1, \dots, \widehat{H}_A$ . Then, the isolating constants  $H_0(\phi), \dots, H_A(\phi)$  turn out to be the only reliable characters in the quantum or the classical play, and the only candidates for thermodynamic variables.

This point can be made in a different way. In the coordinates  $J_{\phi_i}$ , states can only be *locally diagonalized*. For this reason, it is convenient to consider that the unitary operator  $U$  of eq.(51) diagonalizes only the indices of the  $H$  (that we will call  $r$ ) and does not diagonalize the indices of the  $J$  (that we will call  $m$ ). Then, we obtain a basis where the coordinates of the states read (see (52))

$$\rho(\omega)_{\phi_i r m r' m'} = \rho_{\phi_i r m m'}(\omega) \delta_{r r'} \quad (100)$$

With this basis, eqs.(53) and (54) become

$$\widehat{\rho}_* = W \lim_{t \rightarrow \infty} \widehat{\rho}(t) = \sum_{i r m m'} \int d\omega \rho_{\phi_i r m m'}(\omega) (\omega, r m r m' |_{\phi_i} \quad (101)$$

and

$$\widehat{P}_I = \sum_{i r m m'} \int d\omega P_{\phi_i r m m'}^I |\omega r m\rangle_{\phi_i} \langle \omega r m'|_{\phi_i}, \quad I = 1, 2, \dots, A \quad (102)$$

and so on for the rest of the equations. But the  $m$  indices can be 'traced away' because the corresponding observables cannot be measured, i.e. they cannot be considered classical in a global way. This strategy amounts to consider that the  $J$  operators and the  $m$  indices are inexistent in space  $\widehat{\mathcal{O}}$ , to the extent that  $\widehat{\mathcal{O}}$  is the space of all measurable observables. This fact is manifested by the following change in the observables of eq.(40)

$$O(\omega)_{\phi_0 r m r' m'} \rightarrow O(\omega)_{\phi_0 r r'} \delta_{m m'} \quad (103)$$

where we only consider the diagonal term since we are only concerned with the classical part. As we can see, the  $m$  indices have a 'spherical symmetry' and they measure nothing. Moreover, with variables  $H$  and their conjugate configuration variables, we can define a  $2(A + 1)$  manifold  $\phi_0$ , and with variables  $J$  and their conjugated configuration variables, a  $2(N - A)$  manifold  $\phi_{irre}$ , in such a way that  $\mathcal{M} = \phi_0 \otimes \phi_{irre}$ . Then,  $O(\omega)_{\phi_0 r r'}$  is globally defined in  $\phi_0$  and

$$\begin{aligned} \widehat{\mathcal{O}}_S &= \sum_{i r m r' m'} \int d\omega O(\omega)_{\phi_0 r r'} \delta_{m m'} |\omega r m r' m'\rangle_{\phi_i} = \\ &= \sum_{r r'} \int d\omega O(\omega)_{\phi_0 r r'} \sum_{i m} |\omega r m r' m\rangle_{\phi_i} = \sum_{r r'} \int d\omega O(\omega)_{\phi_0 r r'} |\omega r r'\rangle_{\phi_0} \end{aligned}$$

where

$$|\omega r r'\rangle_{\phi_0} = \sum_{i m} |\omega r m r' m\rangle_{\phi_i}$$

is the basis of the new space of observables. Then, the relevant part of eq.(40) reads

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<sup>41</sup>We can only measure dynamical variables when they can be considered as constants in time, at least in the period of measurement. If a constant is not global, it is only constant in time in a local coordinate system, i.e. it is not a usual 'physical constant'.



$$\begin{aligned} \langle \widehat{O} \rangle_{\widehat{\rho}_*^T} &= \sum_{irmr'm'} \int d\omega \overline{\rho(\omega)_{\phi_{irmr'm'}}} O(\omega)_{\phi_{irmr'm'}} = \sum_{irmr'm'} \int d\omega \overline{\rho_{\phi_{irmm'}}(\omega) \delta_{rr'}} O(\omega)_{\phi_{0rr'}} \delta_{mm'} = \\ &= \sum_r \int d\omega \left( \sum_{im} \overline{\rho(\omega)_{\phi_{irmm}}} \right) O(\omega)_{\phi_{0rr}} \end{aligned} \quad (104)$$

Calling  $\sum_{im} \overline{\rho(\omega)_{\phi_{irmm}}} = \overline{\rho(\omega)_{\phi_{0r}}}$ , namely, making the  $m$ -trace of  $\rho(\omega)_{\phi_{0rmr'm}}$ , we obtain the 'traced' equation

$$\langle \widehat{O} \rangle_{\widehat{\rho}_*^T} = \sum_r \int d\omega \overline{\rho(\omega)_{\phi_{0r}}} O(\omega)_{\phi_{0rr}} \quad (105)$$

where the  $m$  indices have disappeared; eq.(101) now reads

$$\widehat{\rho}_*^T = W \lim_{t \rightarrow \infty} \widehat{\rho}(t)^T = \sum_r \int d\omega \rho(\omega)_{\phi_{0r}}(\omega, r|_{\phi_0})$$

where  $T$  now means 'm-traced' and  $\{(\omega, r|_{\phi_0})\}$  is the dual basis of  $\{|\omega, r\rangle_{\phi_0}\}$ . If we work only with the  $r$  indices ( $r_0, \dots, r_A$ ), the  $\rho$  of eq.(96) becomes the  $\rho$  of eq.(99) solving the problem.<sup>42</sup> After this coarse-graining that retains only the  $r$  variables, the relevant phase space  $\phi_0$  has only  $2(A+1)$  dimensions. So, we can repeat all the theory with only these dimensions and find that, at each manifold  $\mathcal{M}(p_0, p_1, \dots, p_A)$ , there is only one constant equilibrium state  $\rho_*^T(\phi)$  given by eq.(99) with  $\rho(p_0, p_1, \dots, p_A) = \rho(\omega)_{\phi_{0r}}$  ( $\omega = p_0, r = (p_1, \dots, p_A)$ ).

The above conclusions are important for two reasons:

1.- They prove that a system with a continuous evolution spectrum is ergodic in  $\mathcal{M}(p_0, p_1, \dots, p_A)$  with a constant equilibrium density  $\rho_*^T$ . Therefore, such a system satisfies the two basic hypotheses of classical statistical mechanics: it is *ergodic* and *microcanonical*.

2.- All the above classical properties can be transferred to the quantum level; therefore, we can define *quantum ergodic systems* as those that satisfy these transferred properties. This definition would be one of the possible definitions of quantum chaos (see [61]): precisely the one which says that a quantum system is chaotic if it has a chaotic classical limit ([45], [47]). This fact is a consequence of our definition of the classical limit.

### C. Non-integrable systems

Let us now consider the particular case of a non-integrable system where only  $H(\phi)$  is globally defined ( $A = 0$ ). Then, we can trace away all the  $J_{\phi_i}$ : the resulting traced system has a global CSCO  $\{\widehat{H}\}$ . As a consequence, in each energy manifold  $H = \omega = \text{const.}$  the motion is ergodic: this is the typical ergodic motion (the drop in the glass of water, etc.). In this case, after tracing away the  $J_{\phi_i}$  (i.e. the  $r$ -coordinates disappear), the pointer basis only depends on  $H$ , namely, it is just the eigenbasis of  $H$ .

## VII. CONCLUSIONS

We want to conclude the paper proposing some suggestions for future research.

i.- We have essentially presented a *minimal formalism for quantum chaos*, to the extent that our quantum formalism satisfies a minimal requirement for such a theory: by definition, a quantum chaotic system has, at least, a classical non-integrable system as its classical limit. In fact, this is a necessary but not a sufficient condition that any proposed theory of quantum chaos must fulfil. Therefore, our next task is to address the question of whether the set of phenomena known under the name of 'quantum chaos' ([45], [47], [51], [55], [61]) can be explained by means of our theoretical structure.

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<sup>42</sup>At this point, we could say that we are working with a *system* given by the  $\widehat{H}_0, \widehat{H}_1, \dots, \widehat{H}_A$  and an *environment* given by the  $\widehat{J}_{\phi_0 1}, \dots, \widehat{J}_{N-A\phi_0}$ : we could consider this situation as the case of an *open quantum system*.

ii.- Quantum contexts are clearly related with  $N + 1$ -CSCOs. We have seen that generic  $N + 1$ -CSCOs are local. This might have a relation with well known physical questions, as the EPR problem and the Kochen-Speker and Bell theorems (see [52]), where paradoxes arise when we try to describe the quantum system with just one CSCO.

iii.- In some sense, the equations of quantum physics have a local character ([62], [63], [64]); we have found that this is also the case of the CSCOs: it might be useful to explore this analogy.

iv.- In quantum gravity we have to combine concepts of a local classical theory, general relativity, with those of a global one, quantum mechanics, and this task has shown to be almost impossible. Now that we have understood the local nature of quantum mechanics, the task might become easier. Or rephrased in a more modest way: in quantum field theory in curved space time, we have the unsolved vacuum definition problem ([65], [66], [67]), where the vacuum (a clearly local classical general relativity notion) is considered on the global grounds of the quantum field theory. Perhaps the present technique might contribute to solve the problem.

### VIII. ACKNOWLEDGMENTS.

The authors are very grateful to Roland Omn  s for several interesting comments, criticisms, and general informations. This paper was partially supported by grants of the Buenos Aires University, the National Research Council (CONICET), and the National Research Agency (FONCYT) of Argentina.

### APPENDIX A: INTEGRABILITY OF THE DIFFERENTIAL EQUATIONS (19) AND (23)

a.- Let us consider eq.(19), first as a mathematical partial differential equation, and then from a physical point of view. Calling  $\frac{\partial H}{\partial p_{qj}} = A_j$ ,  $\frac{\partial H}{\partial q_j} = -B_j$ , eq.(19) reads

$$\sum_{j=1}^N A_j \frac{\partial O_I}{\partial q_j} + B_j \frac{\partial O_I}{\partial p_{qj}} = 0 \quad (A1)$$

where functions  $A_j$ ,  $B_j$  are known. To solve this equation around a hypersurface  $\mathcal{D}$  (containing a point  $\phi_i$ , that we will call  $(q_0^{(0)}, \varphi^{(0)})$ ) is equivalent to find the trajectories crossing  $\mathcal{D}$  such that

$$\frac{dq_0}{A_0} = \dots = \frac{dq_N}{A_N} = \frac{dp_{q0}}{B_0} = \dots = \frac{dp_{qN}}{B_N} \quad (A2)$$

This requires to solve the system of ordinary differential equations

$$\begin{aligned} \frac{dq_1}{dq_0} &= \frac{A_1}{A_0} = a_1, \dots, \frac{dq_N}{dq_0} = \frac{A_N}{A_0} = a_N \\ \frac{dp_{q0}}{dq_0} &= \frac{B_0}{A_0} = b_0, \dots, \frac{dp_{qN}}{dq_0} = \frac{B_N}{A_0} = b_N \end{aligned} \quad (A3)$$

where we have taken  $q_0$  as a parameter. Calling  $\varphi = \{\varphi_\alpha\} = \{q_1, \dots, q_N, p_{q0}, \dots, p_{qN}\}$ , the Lipschitz conditions on functions  $a$  and  $b$  are satisfied if there exists a number  $M > 0$  such that

$$\begin{aligned} |a(q_0, \varphi') - a(q_0, \varphi)| &\leq M \sum_{\alpha=1}^{2N+1} |\varphi'_\alpha - \varphi_\alpha| \\ |b(q_0, \varphi') - b(q_0, \varphi)| &\leq M \sum_{\alpha=1}^{2N+1} |\varphi'_\alpha - \varphi_\alpha| \end{aligned} \quad (A4)$$

If functions  $a$  and  $b$  are continuous and satisfy these conditions in a neighborhood of a point  $(q_0^{(0)}, \varphi^{(0)})$  considered as the initial conditions, then there exists a unique solution of the system (A3) in a neighborhood of  $\varphi^{(0)}$  and for  $q_0 \in [q_0^{(0)} - c; q_0^{(0)} + c]$  for some  $c > 0$ . Moreover, if also

$$\Delta = \begin{vmatrix} A_j & B_j \\ \frac{\partial q_j}{\partial \Phi_j} & \frac{\partial p_j}{\partial \Phi_j} \end{vmatrix} \neq 0$$

where the  $\Phi_j$  are  $q_0$  and the  $2N + 1$  parameters  $\varphi_\alpha$  on the surface  $\mathcal{D}$  (see [46]), then we will also find a local solution of the partial-differential equation (A1).

b.- Let us now consider eq.(19) from a physical point of view, in order to see if the Lipschitz conditions are satisfied. Introducing the time  $t = q_0$  and, therefore,  $p_0 = H$  independent of  $t$ , eqs.(A3) read

$$\frac{dq_i}{dt} = \frac{\partial H(q, p)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H(q, p)}{\partial q_i} \quad (\text{A5})$$

namely, the Hamilton equations. This system is Lipschitzian if there is a  $M > 0$  such that

$$\left| \frac{\partial H(Q, P)}{\partial p_i} - \frac{\partial H(q, p)}{\partial p_i} \right| < M \left( \sum_i |Q_i - q_i| + |P_i - p_i| \right) \quad (\text{A6})$$

$$\left| \frac{\partial H(Q, P)}{\partial q_i} - \frac{\partial H(q, p)}{\partial q_i} \right| < M \left( \sum_i |Q_i - q_i| + |P_i - p_i| \right)$$

in a certain domain  $\mathcal{D}$  of the phase space. These conditions are fulfilled if the derivatives  $\frac{\partial^2 H(q, p)}{\partial q_i \partial q_j}$ ,  $\frac{\partial^2 H(q, p)}{\partial q_i \partial p_j}$ ,  $\frac{\partial^2 H(q, p)}{\partial p_i \partial p_j}$  are bounded in the domain  $\mathcal{D}$  (see e.g. [68] p.141). We can understand the meaning of this requirement in a physical example. Let us consider the Hamiltonian

$$H(q_1, q_2, p_1, p_2) = \frac{p_1^2}{2} + \frac{q_1^2}{2} + \frac{p_2^2}{2} + \frac{q_2^2}{2} + V(q_1, q_2) \quad (\text{A7})$$

For this particular case, the derivatives of the potential  $\frac{\partial^2 V(q, p)}{\partial q_i \partial q_j}$  must be bounded in  $\mathcal{D}$ . Let us consider the potentials  $V(q_1, q_2) = (q_1 - q_2)^\alpha$ . For  $\alpha \geq 2$ , these potentials have their  $\frac{\partial^2 V(q, p)}{\partial q_i \partial q_j}$  bounded in  $\mathcal{D}$  and the system is Lipschitzian (this is the case in quadratic or polynomial potentials used in many models). For  $\alpha < 2$ , these potentials have their  $\frac{\partial^2 V(q, p)}{\partial q_i \partial q_j}$  not bounded in  $\mathcal{D}$  and the system is not Lipschitzian (this is the case of Newton potential  $(q_1 - q_2)^{-1}$ ). But realistic physical potentials are always bounded, e.g. the Lenard-Jones potential for finite energies. In fact, there is always a central repulsive core. This is not the case of the (mathematical) three bodies problem with Newton potentials, provided that all the planets can get infinitely close; but, of course, this is not a physical situation. In conclusion, all the physical potentials are Lipschitzian.

c.- Let us now consider the solution of eq.(23) in a powers of  $\hbar$  expansion. A simple computation shows that, in this case, the problem reduces, for each term, to the one studied in paragraphs a.- and b.- of this Appendix, but for non-homogeneous equations. In fact,

$$\{O_I(\phi), O_J(\phi)\}_{mb} = \{O_I(\phi), O_J(\phi)\}_{pb} + \hbar^2 A^{(1)}(\phi) + \dots = 0$$

Then, if

$$O_I(\phi) = O_I^{(0)}(\phi) + \hbar^2 O_I^{(1)}(\phi) + \dots$$

we have that

$$\{O_I^{(0)}(\phi), O_J^{(0)}(\phi)\}_{pb} + \hbar^2 \left( \{O_I^{(1)}(\phi), O_J^{(1)}(\phi)\}_{pb} + A^{(1)}(\phi) \right) + \dots = 0$$

so,

$$\{O_I^{(0)}(\phi), O_J^{(0)}(\phi)\}_{pb} = 0,$$

$$\{O_I^{(1)}(\phi), O_J^{(1)}(\phi)\}_{pb} + A^{(1)}(\phi) = 0, \dots$$

namely, the non-homogeneous version of eq.(A1), which can be solved by means of the same method. In this case, the domain of the solution will be the intersection of the domains of solution of all these equations.

## APPENDIX B: SINAI BILLIARD.

Let us consider the Sinai billiard of fig.1 [47]. It is clear that, when the is confined to the inside of the billiard, the trajectories are defined by two independent constants of the motion,  $H$  and  $P_x$  (or  $H$  and  $P_y$ , or  $P_x$  and  $P_y$ ), which constitute a complete set of local (i.e. in the interior  $D_0$  of the billiard) constants of the motion in involution. When the ball strikes the boundaries, it is symmetrically reflected, i.e. the incident angle is equal to the reflected angle, and the value of some of the constants of the motion changes: for the two horizontal boundaries,  $H$  and  $P_x$  still constitute a complete set of local constants of the motion in involution, but  $P_y$  changes its sign; for the vertical boundary,  $H$  and  $P_y$  still constitute a complete set of local constants of the motion in involution, but  $P_x$  changes its sign.

Without modifying the physical characterization of the example, we can replace the rigid walls with infinitely height potential barriers of width  $d$ , namely, the potentials  $V(x)$ ,  $V(y)$  and  $V(r)$  of figure 2. (e.g.  $V(x)$  behaves as  $V(0) = 0$ ,  $V'(0) = 0$ ,  $V(-d) \rightarrow \infty$ ). Due to the symmetry of the potentials (translation symmetry for  $V(x)$  and  $V(y)$ , rotation symmetry for  $V(r)$ ), the reflections are still symmetric, i.e. the ball climbs the potential walls and then falls down with symmetrical motion. Calling  $D_1$  and  $D_3$  the domains in the potential of the  $x$  walls,  $D_2$  that of the  $y$  wall, and  $D_4$  that of the curved wall, we see that  $x$  is a cyclic variable in  $D_1$  and  $D_3$ ,  $y$  is a cyclic variable in  $D_2$ , and  $\theta$  is a cyclic angular variable in  $D_4$ . Therefore, we have the following local constants of the motion in each domain:

$$\begin{array}{lll} D_0 : & H & P_x \quad (\text{or } P_y) \\ D_1 : & H & P_x \\ D_2 : & H & P_y \\ D_3 : & H & P_x \\ D_4 : & H & P_\theta \end{array}$$

In summary, we have found five domains, each one with two constants of the motion in involution. If  $d \rightarrow 0$ , we go from fig.2 to fig.1.

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**Figure Caption:**

fig.1 A Sinai billiard.

fig.2 A Sinai billiard with potential barriers.